Bachelor of Commerce
(B.Com.)

Mathematics and Logical Reasoning
(DBCMDS201T24)

Self-Learning Material
(SEM II)

Jaipur National University
Centre for Distance and Online Education

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Course Introduction

The aims of teaching and learning mathematics are to encourage and enable students to:

- recognize that mathematics permeates the world around us
- appreciate the usefulness, power and beauty of mathematics
- enjoy mathematics and develop patience and persistence when solving problems
- understand and be able to use the language, symbols and notation of mathematics
- develop mathematical curiosity and use inductive and deductive reasoning when solving problems
- become confident in using mathematics to analyse and solve problems both in school and in real-life situations
- develop the knowledge, skills and attitudes necessary to pursue further studies in mathematics
- develop abstract, logical and critical thinking and the ability to reflect critically upon their work and the work of others
- develop a critical appreciation of the use of information and communication technology in mathematics
- appreciate the international dimension of mathematics and its multicultural and historical perspectives.

This course has 4 credits and 14 units. Each unit is divided into sections and sub-sections. Each unit begins with statement of objectives to indicate what we expect you to achieve through the unit.

Course Outcomes

After studying this course, a student will be able to –

1. Recall the basic of mathematics, its concepts & Compound Interest and basics of Logarithms
2. Demonstrate business mathematics concepts that are encountered in the real world, understand and be able to communicate the underlying business concepts and mathematics involved to help another person gain insight into the situation.
3. Use correct mathematical terminology and symbolic processes in order to be prepared for future work in business.
4. Analyse various mathematical models to solve business problems.
5. Assess real world scenarios to recognize when simple and compound interest, annuities, payroll preparation
6. Build critical thinking, modelling, and problem-solving skills in a variety of contexts.We hope you enjoy the course. Please attempt all the assignments and exercises given in the units.

We hope you will enjoy the course.

Acknowledgement

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Unit 1
The Number System

Learning Outcomes:

After studying the unit, students will be able to:

- Increase a suitable understanding of the number system.
- Identify different types of number system
- Convert one type of number system into another such as decimal to binary, octal etc.
- Understand conversion Rules of Number Systems

Structure:

1.1. Introduction

Each and every one, in this world has almost come across the term called as numbers. So what exactly do numbers mean? This is one of the terms which we need to clarify before we go into the depth of number systems. Numbers are just the calculative figures through which it makes the human race, even the other living organisms, to understand the concept of counting and lead their life by understanding the concept of statistical values, in terms of humans, that come up in their lives. Numbers are the infinite calculative fixed figures which give the exact statistical data in terms of the mathematical counting concepts. Now this concept of numbers with reference to the counting systems, which gives us the serial values of numbers, together lets us understand another important term which is related to the number system, which is also known as the number line. After we are clear with the concept of numbers, we move next to another term, called number line.

Number line, as the name defines itself, is nothing but the way and a process of how the numbers can be or are represented on an axis which is either horizontal or vertical, called a line. Horizontal lines can be generally seen in normal calculations that people do in their day to day lives.

Number systems are systems in mathematics that are used to express numbers in various forms and are understood by computers. A number is a mathematical value used for counting and measuring objects, and for performing arithmetic calculations. Numbers have various categories like natural numbers, whole numbers, rational and irrational numbers, and so on. Similarly, there are various types of number systems that have different properties, like the binary number system, the octal number system, the decimal number system, and the hexadecimal number system. The binary number system, the octal number system, the decimal number system, and the hexadecimal number system are only a few examples of the several number systems that humans utilize. To further comprehend the notion, we will learn how to convert between various number systems and solve instances.

1.2. What is a Number System?

A system for representing numbers is called a number system. It also establishes a set of values to represent a quantity and is referred to as the system of numeration. The most often used digits among them are the binary numbers 0 and 1, which are used to represent them. Other sorts of number systems are represented by the digits 0 to 9.

The consistent representation of numbers using digits or other symbols is known as a number system. Any digit in a number can have its value defined by that digit, its place in the number, and the number system's base. Because of the distinctive way the numbers are represented, we can perform mathematical operations like addition, subtraction, and division.

1.2.1. What is a Number?

A number is a numerical value that may be used to count, measure, or name items. The process of completing mathematical calculations uses numbers. Natural numbers, whole numbers, rational and irrational numbers, etc. are some examples of numbers. Additionally, 0 is a number that denotes a null value.

Other variants of a number include prime and composite numbers, even and odd numbers, and many others. In contrast to prime and composite, which distinguish between numbers with only two components and those with more than two factors, even and odd phrases are used to indicate whether or not a number is divisible by two. These numbers are used as digits in a number system. The most frequent digits in the number system used to express binary numbers are 0 and 1.On the other hand, several number systems also employ 0 to 9 digits. On the other hand, some number systems also employ the range of 0 to 9. Learn about the many number systems here. Here, let's examine the various number systems.

1.2.2. Number Line

A Number line is a representation of Numbers with a fixed interval in between on a straight line. A Number line contains all the types of numbers like natural numbers, rationals, Integers, etc.

Numbers on the number line increase while moving Left to Right and decrease while moving from right to left.

1.2.3. Positive Numbers: Numbers that are represented on the right side of the zero are termed as Positive Numbers. The value of these numbers increases on moving towards the right. Positive numbers are used for Addition between numbers. Example: 1, 2, 3, 4, …

1.2.4. Negative Numbers: Numbers that are represented on the left side of the zero are termed as Negative Numbers. The value of these numbers decreases on moving towards the left. Negative numbers are used for Subtraction between numbers. Example: -1, -2, -3, -4, …

1.3. Number and Its Types

A number is a value created by the combination of digits with the help of certain rules. These numbers are used to represent arithmetical quantities. A digit is a symbol from a set of 10 symbols ranging from 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Any combination of digits represents a Number. The size of a Number depends on the count of digits that are used for its creation.

For Example: 123, 124, 0.345, -16, 73, 9, etc

For Example: 123, 124, 0.345, -16, 73, 9, etc

Ends of a number line are not defined i.e., numbers on a number line range from infinity on the left side of the zero to infinity on the right side of the zero.

1.3.1. Natural Numbers

One of the essential types of number system is what we call it, Natural Numbers. Natural Numbers are nothing, but they are just the normally represented numbers which starts from the number one (1) and goes upto the positive infinity. Generally, we all know that the basic number counting system starts from the number zero (0). But one thing we need to keep in mind is that the nu8mbers which are negative do not come under the natural numbers list, as natural numbers hold only the positive values excluding only zero (0). Natural Numbers can be represented with the letter, “N”.

N = 1,2,3,4,5,6,7,8,9,10,11,12,..infinity.

N = { 1,2,3,4,……….}

1.3.2. Whole Numbers

Another type of term which holds its name in the list of number systems is, what we call, the whole numbers. Whole Numbers are nothing but the numbers which start from the number zero (0) and end up to the positive infinity. It is the same case, as in that of the natural numbers, that whole numbers too, only hold its name with the positive numbers. Whole Numbers are represented with the letter, “W”.

W = 0,1,2,3,4,5,6,7,8,9,10,11,12, ...infinity.

W = {0,1,2,3,4, …………}

1.3.3. Integers

We have been seeing that the values which are positive have got their positions and names already, but what about the negative numbers? Of course, we cannot deny the existence of negative numbers, especially in the field of mathematics. Thus, if we are to be told to combine both the positive and negative values then, we get a new term called as Integers. There is a term called “zahlen”, which is originally a word which came from the German language, that means which is used to count. Thus, integers are denoted as, “Z”.

Z = 0, -2,687,9204, -475,294, -9483,1, -3, etc (which includes all the negative and positive integers, except those which are in fractions).

Z = {…...-3, -2, -1,0,1,2, 3……...}

1.3.4 Rational Numbers

Rational Numbers are those numbers which can be written in fractions. In simple meaning, the fractional numbers that were left out from the list of integers, are put under the list of rational numbers. Actually, the word rational has been derived from the word ratio, which means fraction. In a fraction, we all know there is one numerator and one denominator. In the case of rational numbers, it is more specific for a number to be a rational number. In this case, the numerator is considered as "p" and the denominator is considered as "q".

Thus, for a number to be termed as rational number is that the number should be in "p/q" form where the value of the term q should not be equal to zero, "q != 0", or else the value of the number will be turned out to be infinity, as we know that if the denominator in any fraction is zero (0), then the value turns out to be zero (0). Whenever one tries to solve the rational numbers, then there are two possibilities for the results to be out. One of them is that the remainder will be directly zero (0). For example, if we try to solve the fraction 7/8, which means 7 is getting divided by 8, and the result comes out to be 0.875, which means the remainder is zero (0).

The other possibility is that the remainder never turns out to be the value equal to zero (0). For example, if we try to solve the fraction 10/3, which means 10 is getting divided by 3, and the result turns out to be as 3.333333….., which never ends. To make it easy to represent such a number, one can put a small bar over that number which is getting repeated. In this example taken, it can be written as 10/3 = 3.3 bar.

The numbers which, like in the second case, where the remainder never ends and never turns out to be zero are called non-terminating and recurring, where non-terminating means never ending and recurring means repeating. In some cases, the rational numbers are also termed as terminating and recurring.

Rational Numbers are represented as the letter, "Q".

Rational Numbers can also be written as

Q = { p/q, p , q € Z , q $\neq 0$ } (€ -means belongs to)

1.3.5. Irrational Numbers

Here, as the name suggests, Irrational numbers are simply those numbers, which cannot be represented in the way a rational number is represented. It means, the numbers, which when are in fraction or without fraction, are not in the form of p/q, means the numerator is not in the form of p and q is not the form of denominator. For example, the numbers which are under the square roots or cube roots, that cannot be solved further to find its roots are called Irrational Numbers. Irrational numbers also include those numbers which are in fraction and when solved, gives us a value which never ends and also never gets repeated, such a number is called non-terminating and non- recurring.

For example: - value of pie which is equal to 3.14………., square root of 7 ($\sqrt{7}$), square root of 8 ($\sqrt{8}$), cube root of 6 ($\sqrt[3]{6}$), etc.

1.3.6. Real Numbers

These numbers are nothing but simple the whole total combination of the rational numbers with addition the the irrational numbers. Real Numbers are represented with the letter, "R". These numbers include whole numbers, integers, fractions, etc. All the integers belong to Real numbers but all the real numbers do not belong to the integers.

N⊆ W ⊆ Z ⊆ Q ⊆ R (where ⊆ means subset of) *⊆ - will see in second Unit

1.3.7. Imaginary Numbers

Imaginary Numbers are all those numbers that are not real numbers. These numbers when squared will result in a negative number. The √-1 is represented as i. These numbers are also called complex numbers.

1.3.8. Prime Numbers and Composite Numbers

Numbers that do not have any factors other than 1 and the number itself are termed as Prime Numbers. All the numbers other than Prime Numbers are termed as Composite Numbers except 0. Zero is neither prime nor a composite number.

Example:

1) 2,3,5,7,11,13 etc are prime number
2) 6,18, 25, 40 etc are composite numbers.

1.3.9. Fractions

Fractions are the numbers that are written in the form of a/b, where, a belongs to Whole numbers and b belongs to Natural Numbers, i.e., b can never be 0. The upper part of the fraction i.e. a is termed as a Numerator whereas the lower part i.e. b is called Denominator.

Example: 2/3 , 7/5 11/17 are fractions.

1.4. Types of Number System

In mathematics, there are many different kinds of number systems. The four most typical varieties of number systems are:

1 System of decimal numbers (Base- 10)

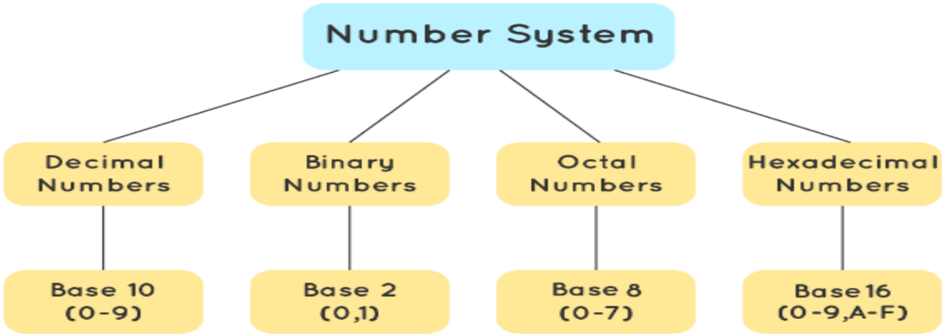
2 System of binary numbers (Base- 2)

3 System of octal numbers (Base-8)

4 Hexadecimal system of numbers (Base- 16)

1.4.1. Number System Chart

Given below is a chart of the main four types of number system that we use to represent numbers.



1.4.2. Binary Number System

Only the digits 0 and 1 are used in the binary number system. The basis of the numerals in this system is two. Bits are the digits 0 and 1, and a byte is made up of 8 bits. Computers store information in terms of bits and bytes. Other numbers, such 2,3,4,5, and so on, are not included in the binary number system. The binary number system, for instance, uses the following instances of numbers: 100012, 1111012, and 10101012.The term "Octal Number System" refers to a system of numbers with an eight-digit base with digits ranging from 0 to 7. The term "octal" is used to describe numbers with an eight base. The octal numbers are important and have various uses, including in digital numbering systems and computers.

By first converting a binary number to a decimal number and then a decimal number to an octal number, octal numbers may be translated to binary numbers and binary numbers to octal numbers in the number system.

Example:

Write $(14)_{10}$ as a binary number.

2	14	
2	7	0
2	3	1
	1	1

↑

∴ $(14)_{10} = 1110_2$

1.4.3. Octal Number System

The octal number system employs eight numbers with a base of eight: 0, 1, 2, 3, 4, 5, 6, and 7. The benefit of this system is that there will be less computing mistakes because it has fewer digits than many other systems. The octal number system does not include digits like 8 and 9. In minicomputers, the octal number system, which uses digits from 0 to 7, is employed similarly to the binary system. Examples of numbers in the octal number system are 358, 238 and 1418. The term "Octal Number System" refers to a system of numbers with an eight-digit base with digits ranging from 0 to 7. The term "octal" is used to describe numbers with an eight base. The octal numbers are important and have various uses, including in digital numbering systems and computers. By first converting a binary number to a decimal number and then a decimal number to an octal number, octal numbers may be translated to binary numbers and binary numbers to octal numbers in the number system.

Example: Convert 215_8 into decimal.

Solution:

$215_8 = 2 \times 8^2 + 1 \times 8^1 + 5 \times 8^0$

$= 2 \times 64 + 1 \times 8 + 5 \times 1$

$= 128 + 8 + 5$

$= 141_{10}$

1.4.4. Decimal Number System

The base number of the decimal number system is 10, while the other nine digits are 0, 1, 2, 3, 5, 6, 7, 8, and 9. In everyday life, we often express numbers using the decimal number system. Any number's base is 10 if it is expressed without a base. The decimal number system, for instance, uses the numbers 72310, 3210, and 425710. Our daily usage of numbers is based on a decimal number system, which uses 10 digits. A number system in mathematics is thought of as the use of digits or symbols to represent numbers. The binary number system, the decimal number system, the octal number system, and the hexadecimal number system are the four primary varieties of the number system. Given that it was challenging to multiply and divide big numbers by hand in earlier civilizations, the decimal number system is sometimes known as the Hindu-Arabic or Arabic number system. Let's explore the decimal number system in greater detail. The digits used in the decimal number system, which we use every day, are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Since there are a total of 10 possible numbers in the decimal number system, 10 serves as the base number for this system. Any number's base is 10 if it is expressed without a base.

Example of Decimal Number System:

The decimal number 1457, whose value is represented as:, has the digits 7 in the units place, 5 in the tens place, 4 in the hundreds place, and 1 in the thousands place.

$(1\times10^3) + (4\times10^2) + (5\times10^1) + (7\times10^0)$

$(1\times1000) + (4\times100) + (5\times10) + (7\times1)$

1000 + 400 + 50 + 7

1457

1.4.5. Hexadecimal Number System

The letters "Hexa" and "deci" in the phrase "hexadecimal" stand for the numbers "6" and "10," respectively. The 16-digit hexadecimal representation of numerals 0 through 9 and letters A through F is known as the hexadecimal number system. To put it another way, the first nine numbers or digits are represented as numbers, while the next six digits are represented by the letters A through F. The decimal number system, which has a base of 9, and hexadecimal are quite similar.

As a result, after 9 digits, the 10th digit is represented by a symbol: 10, 11, 12, 13, 14, and 15, respectively. A, B, C, D, E, and F are the 16 numbers as a result of 1, 2, 3, 4, 5, 6, 7, 8, and 9. The hexadecimal integers 7 B 3 16; 6 F 16; and 4 B 2 A 16 are examples.

Because each digit has a weight of power 16, the hexadecimal number system is also known as a positional number system. Each subsequent digit is 16 times more important than the one before it. As a result, whenever we convert a hexadecimal number to another number system, we multiply each digit separately while taking into account the power of 16 and the location of each digit. The base number of the hexadecimal number system is 16, and there are sixteen digits/alphabets: 0, 1, 2, 3, 5, 6, 7, 8, 9, and A, B, C, D, E, F. Hexadecimal letters A through F here correspond to the decimal numerals 10-15, respectively. Computers employ this method to compress the lengthy binary system strings. Examples of numbers in the hexadecimal number system are 7B316, 6F16, and 4B2A16.

Example: Convert hexadecimal (76)16(76)16 to binary.

Solution: Convert (76)16(76)16 to decimal by multiplying each digit with 16^{n-1}. Multiply it

(76)16(76)16 = $7 \times 16^{(2-1)} + 6 \times 16^{(1-1)}$

(76)16(76)16 = $7 \times 16^{1} + 6 \times 16^{0}$

(76)16(76)16 = $7 \times 16 + 6 \times 1$

(76)16(76)16 = 112 + 6

(76)16(76)16 = 118

Therefore, (76)16(76)16 = (118)10(118)10.

Convert (118)10(118)10 to a binary number by dividing the number by 2 until the quotient is zero.

1.4.6. Conversion Rules of Number Systems

Using formulae for different number systems, a number can be changed from one number system to another. Octal numbers can be converted to decimal numbers and vice versa, just as binary numbers can be converted to octal numbers and vice versa. Let's examine the procedures needed to change number systems. Any kind of number system, including binary, decimal, hexadecimal, etc., can be used to represent a number.

Additionally, it is simple to change any number that is represented in one number system type into another. To understand how to convert numbers in decimal to binary and vice versa, hexadecimal to binary and vice versa, and octal to binary and vice versa using numerous examples, check out the in-depth course on the conversions of number systems. Let's now briefly explore the conversion of one number system to another by selecting a random number using the various conversion processes that were previously taught.

Assume the number 349. Thus, the number 349 in different number systems is as follows:

The number 349 in the binary number system is 101011101

The number 349 in the decimal number system is 349.

The number 349 in the octal number system is 535.

The number 349 in the hexadecimal number system is 15D

1.5. Steps for Conversion of Binary to Decimal Number System

The procedures below are used to translate a number from the binary to decimal systems.

Step 1 : First, multiply each digit of the provided number by the exponents of the base, starting with the digit on the right.

Step 2: As we progress from right to left, the exponents should begin at 0 and rise by 1.

Step 3: Condense each of the aforementioned items and include them.

1.5.1. Conversion of Decimal Number System to Binary / Octal / Hexadecimal Number System

The procedures below are used to translate a number from the decimal number system to the binary, octal, or hexadecimal number systems. The process for changing a number from the decimal system to the octal system is demonstrated.

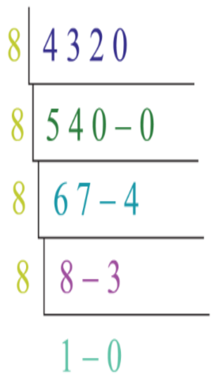
For instance, change 432010 to the octal system.

Solution:

Determine the basis of the necessary number in

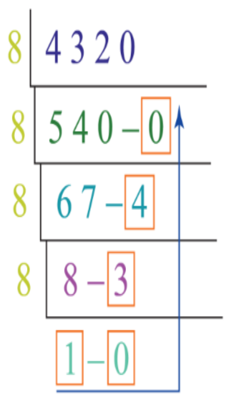
Step 1. The base of the necessary number is 8, as we must convert the provided number into the octal system.

Step 2: Calculate the quotient and remainder in the quotient-remainder form by dividing the provided number by the base of the necessary number.



Divide the quotient by the base once again, repeating this step until the quotient equals the base.

Step 3: To get the supplied number in the octal number system, read the final quotient and all remainders from bottom to top.



Therefore, $4320_{10} = 10340_{8}$

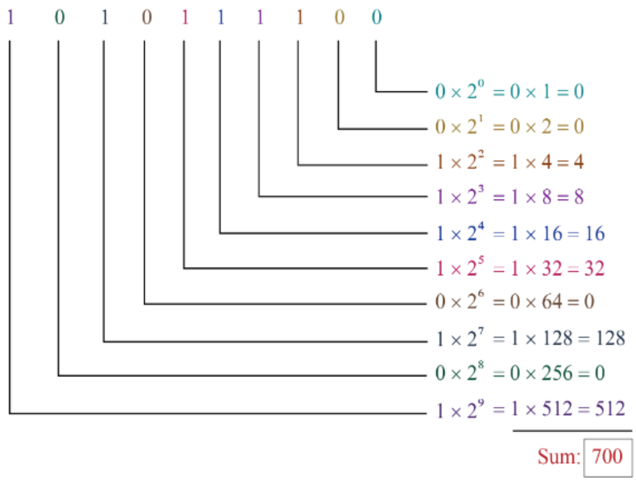
1.5.2. Conversion from One Number System to Another Number System

The above-mentioned procedures are used to convert a number from one of the binary, octal, or hexadecimal systems to one of the other systems after first converting it to the decimal system.

Example: Convert 1010111100_2 to the hexadecimal system.

Solution:

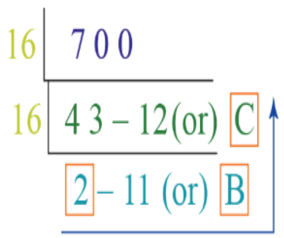
Step 1: Convert this number to the decimal number system as explained in the above process.



Thus, $1010111100_2 = 700_{10} \rightarrow (1)$

Step 2: Convert the above number (which is in the decimal system), into the required number system (hexadecimal).

Here, we have to convert 700_{10} into the hexadecimal system using the above-mentioned process. It should be noted that in the hexadecimal system, the numbers 11 and 12 are written as B and C respectively.



Thus, $700_{10} = 2BC_{16} \rightarrow (2)$

From the equations (1) and (2), $1010111100_2 = 2BC_{16}$

1.6. Summary

In this Unit, we have studied the following points

Number and types

- ✔ Natural number N= {1,2,3, 4………}
- ✔ Whole number W = {0,1,2,3, 4……}
- ✔ Integers Z = {……. -3, -2, -1,0,1,2,3……}
- ✔ Rational numbers Q = {p/q, p. q € Z, q ≠ 0}
- ✔ Real Number N⊆ W ⊆ Z ⊆ Q ⊆ R

Number System

- ✔ Decimal number- 0,1,2,3,4,5,6,7,8,9
- ✔ Hexadecimal number -0,1,2,3,4,5,6,7,8,9, A, B, C, D, E
- ✔ Octal number -0,1,2,3,4,5,6,7

1.7. Self-assessment questions

1. Which of the following are not natural numbers?

   a) -2 b) 3 c) 7 d) 113 e) 5/6 f) 4.5
2. Which are the integersof the following?

   a) 8 b) -7 c) 3 /4 d)220 e) -8.5 f) 13
3. Write first 15 prime numbers.
4. Which of the following are rational numbers?

   a) 7/9 b) -7 c) 3 /4 d)220 e) – 13/17 f) 3345
5. Check whether 156 is a composite number or not, if so, write its factor.
6. Convert decimal 450 number to binary.
7. Convert 110011001 to decimal.
8. If this number is binary 101, if so, convert to octal.
9. Convert octal 67 to hexadecimal number.
10. 1Write 10,11,12,13,14,15 in octal and hexadecimal numbers.

1.8. Reference

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- Apostol, T. (1991): Calculus, Volumes 1 and 2, Wiley.
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Unit 2

Simple And Compound Interest

Learning Outcomes:

After studying the unit, students will be able to:

- Concept of simple Interest.
- Concept of compound interest
- Comparison of simple and compound interest.

Structure:

2.1. Introduction

2.2. Basic of Simple interests

2.3. Compound interest

2.4. Example Using simple and Compound Interest

2.5. Difference between Simple interest and compound interest

2.6. Summary

2.7. Self-assessment questions

2.8 Reference

2.1. Introduction

Simple interest is often a predetermined percentage of the principle amount borrowed or lent paid or received over a specific time period. Borrowers are required to pay interest on interest in addition to principal since compound interest accrues and is added to the accrued interest from prior periods.

Interest is determined based on the loan or investment made. Interest can be calculated in two different methods. Simple interest (SI) and compound interest are the two methods (CI). Basically, simple interest is the cost of a loan or investment. It is determined using the principal sum. The interest that is calculated on interest is known as compound interest. It is based on both the principal sum and the interest from the prior period. The distinction between simple interest and compound interest is thoroughly explained in this article.

Simple interest is based on the principal amount, which is the main distinction between it and compound interest. Contrarily, compound interest is calculated using the principle sum and interest that has been compounded over the course of a period. We are aware that banking in particular uses both simple interest and compound interest, which are two crucial ideas. Simple interest is used in loans including mortgages, auto loans, student loans, and installment loans. The majority of savings accounts pay interest through compound interest. It offers benefits beyond just interest. Let’s go into more detail about the distinction between simple interest and compound interest in this article.

Simple Interest: The principal amount of a loan or deposit made into a person’s bank account is considered simple interest.

Compound Interest: Interest that accrues and compounds over the principal sum is known as compound interest.

2.2. Basic of Simple interests

The price of borrowing is known as simple interest (SI).It represents the interest on the principal solely, expressed as a percentage of the principal. Because they only have to pay interest on loans they take out, borrowers will gain from simple interest. Simply put, simple interest is the sum paid to the borrower for using the borrowed funds for a predetermined amount of time. By multiplying the interest amount by the tenure and the principal amount, one may quickly calculate simple interest. Simple interest does not take prior interest into account. It is based solely on the initial contribution sum.

When calculating the interest payments on consumer and auto loans, basic interest is used. Simple interest is used to determine the investment return on even certificates of deposit. Since compounding has no effect, simple interest is more advantageous to borrowers. This means that there isn’t interest on interest. However, if investments are based on basic interest, investors could lose money.

Formula for Simple interest

Simple Interest= P*T*R /100

P is the principal amount

R = Period Interest Rate

T = Tenure

2.3. Compound interest

Compound interest (CI), in contrast to simple interest, gains interest on both the principal and any prior interest that has been earned. The principal amount is increased by the interest. Interest on Interest is referred to as CI. The entire idea is to maximise profits by compounding the interest earned on the original amount. In other words, CI offers the chance to generate a greater return than just plain interest on an investment. Compound interest, which is based on the primary power of compounding, causes investments to increase in value exponentially.

The frequency of compounding is determined by the bank, financial institution, or lender. It might occur daily, weekly, biweekly, quarterly, or yearly. The amount of interest accrual will increase with the frequency of compounding. Compound interest therefore favours investors more than borrowers.

For some loans, banks use compound interest. However, investments are where compound interest is most frequently applied. Additionally, fixed deposits, mutual funds, and any other investment that allows for the reinvestment of earnings all use compound interest.

Formula for compound interest

C.I. = $P(1+R/100)^t - P$

CI –Compound Interest is .

P is for the principal amount,

R is the interest rate,

t is the number of years (duration)

2.4. Example Using simple and Compound Interest

Question 1: A sum of Rs 5000 is borrowed and the rate is 5%. What is the simple and compound interest for 2 years?

Solution:

Simple Interest = Principle × Rate × Time = PTR/100

⇒ Simple Interest = 5000 × (5 ⁄ 100) × 2

⇒ Simple Interest = 500

∴The simple Interest for 2 years is Rs. 500

Compound Interest = Principal × (1 + Rate)^{Time} – Principal

So, Compound Interest = 4000 × (1 + 5 ⁄ 100)^2 – 4000

⇒ Compound Interest = (4000 × 1.1025) – 4000

⇒ Compound Interest = 410

∴The compound interest for 2 years is Rs. 410

Question 2: A sum of Rs. 2000 becomes Rs. 5000 at the end of 3 years when calculated at simple interest. Find the rate of interest.

Solution:

Given,

Principal = P = Rs. 2000

Time = T = 3 years

Amount at the end of 3 years = Rs. 5000

SI = Rs. 5000 – Rs. 2000 = Rs. 3000

SI = PTR / 100

⇒ R = SI × 100 / PT

⇒ R = 3000 × 100 /(2000 × 3)

⇒ R = 50%

Hence, the rate of interest = 50%

Question 3: Find the compound interest on Rs. 10000 at 10% for 3 years, compounded annually.

Solution:

Given,

Principal= P = Rs.10000

Rate of interest= r= 10%

Time=t=3 years

Amount on CI = P(1 + r/100)^2

= 10000(1 + 10/100)^3

= 10000 (1 + 0.1)^3

= 10000(1.1)^2

= 10000 × 1.21

= 12100

CI = Amount on CI – Principal

= Rs. 12100 – Rs. 10000

= Rs. 2100

Therefore, the compound interest = Rs. 2100

2.5. Difference between Simple interest and compound interest

Simple Interest and Compound Interest Differences

Parameter	Simple Interest	Compound Interest
Definition	Simple Interest is the amount repaid for using the borrowed funds over a predetermined amount of time.	When the total principal amount surpasses the payment due date and the rate of interest for a certain amount of time, it is known as compound interest..
Formula	$S.I. = (P \times T \times R)/100$	$C.I. = P(1+R/100)^t - P$
Return Amount	When compared to compound interest, the return is significantly smaller.	The payout is substantially more.
Principal Amount	The principal sum never changes.	Throughout the entire loan duration, the principle amount changes.
Growth	This strategy maintains a fairly consistent growth.	With this strategy, the growth increases pretty quickly.
Interest Charged	The principal amount is what the interest is based on.	Its principal and accrued interest is covered by the interest that is imposed on it.

2.6. Summary

- ✔ Simple Interest- The principal amount of a loan or deposit made into a person’s bank account is considered simple interest.
- ✔ Simple Interest= P*T*R /100
- ✔ Compound Interest: Interest that accrues and compounds over the principal sum is known as compound interest.
- ✔ C.I. = $P(1+R/100)^t - P$

2.7. Self-Assessment questions

1. What is Simple Interest?
2. What is compound Interest?
3. Write difference between simple and compound interest
4. Find the compound interest on Rs. 20000 at 15% for 4 years, compounded annually.
5. A sum of Rs. 20000 becomes Rs. 30000 at the end of 2 years when calculated at simple interest. Find the rate of interest.

2.8. Reference

- ✔ https://granite.pressbooks.pub/math502/Unit/simple-and-compound-interest/
- ✔ https://scripbox.com/fd/difference-between-simple-interest-and-compound-interest/

Unit 3

Discount And Annuity

Learning Objectives

After studying the unit, students will be able to:

- Concept of discount
- Rate of discount
- Use of annuity
- Fundamental of annuity
- Types of annuity

Structure

3.1. Discount

3.2. Rate of Discount

3.3. Discount Rate

3.4. Annuities

3.5. Fundamentals of Annuities

3.6. Types of Annuities

3.7. Summary

3,8. Self-assessment questions

3.9. Reference

3.1. Discount

All of us are eager to take advantage of discounts, but we hardly understand how to do so. A discount formula is used to compute discounts in the right way. The selling price of the item can be deducted from its marked price to obtain the discount rate formula, or the marked price of the item can be multiplied with the provided discount rate. The formula for discount is shown below in mathematical language.

Formula

Discount = Marked Price – Selling Price

Discount Percentage Formula = Marked Price × Discount Rate

- Other basis Discount formula are as below: -

Discount = List Price - Selling Price

Definition of Discount

The pricing scheme in which the cost of a thing (or service) is less than its marked price is referred to as a "discount." Simply put, a discount represents a percentage off the advertised price. In consumer interactions, a discount is a type of product price reduction where customers have suggested a percentage of rebates on various goods in order to increase sales. The discount that the seller is providing to the buyer is known as a rebate.

The discount is always calculated on the listed price while considering the selling price.

'Listed price' - the usual price of a commodity not inclusive of any discount.

'Selling price' - the amount we actually pay to acquire the commodity when we buy.

3.2. Rate of Discount

When an item's price is lowered and it is purchased, a discount has been provided. The price reduction expressed as a percentage is referred to as a "discount percentage" or "discount rate." The following formula is used to determine the discount rate:

Discount (percentage) = (List Price - Selling Price)/ List Price x 100

Discount % = (Discount/List Price) ×100

Rate of Discount=Discount % = (Discount/List Price) ×100

Successive Discounts

When two or more discounts are provided one after the other, they are referred to as sequential discounts or discounts in sequence. Having said that, let's say a 15% discount is offered on a purchase. A further 12% discount is then applied to the product's lowered price. We refer to this situation as repeated discounts of 15% and 12% being granted.

Example 1.

Find a single discount equivalent to two successive discounts of 20% and 15% on an article.

Solution:

Let the listed price of an article be Rs 100.

Then, first discount on it = 20%

Price after the 1st discount = Rs (100 - 20) = Rs.80

2nd discount on the updated price i.e. = 15% of Rs.80

= 15/100 *80 =12.75

Thus, Price after 2nd discount = Rs (80 – 12.75) = Rs. 67.25

Net selling price = 67.25

Single discount equivalent to offered successive discounts = (100 – 67.25) % = 32.75%.

Example 2

In an apparel store, a sweatshirt that sells for Rs1000 is marked "15% off." Find out the rate of discount? What is the sale price of the sweatshirt?

Solution:

Assessment: Taking into account that the clothing store frequently offers products at a discount. In essence, a vendor will reduce the price of an item by a certain percentage of the list price. In this case, a product that originally costs Rs1000 is being discounted by 15%. So "15% off" refers to the percentage of the discount. We need to follow a process in order to tackle this challenge.

Procedure

The rate is generally given as a percent.

To calculate the discount, just multiply the rate by the original price.

To compute the sale price, deduct the discount from the original price.

Now that we have a procedure in place, we can simply solve the above problem

Solution: Given that the rate is 15%.

The discount is: 0.15 x Rs1000 = Rs150

Now, the sale price is reckoned as follows:

Original price = Rs1000

Discount – 150

Thus, the sale price = Rs 850

Example 3

A dealer purchased a Fridge for Rs15000. He offered a discount of 15% on its listed price and still gained 10%. Find the listed price of the Fridge.

Solution

Cost price of the oven = Rs15000, Profit% = 10%.

Thus, selling price = {100 + profit %} /100 × CP

= Rs (100+10)100× 15000

= Rs 110/100 × 15000

= Rs 15000.

Now, Let the listed price be Rs x.

Then, the discount = 15 % of Rs. X

= Rs {x × (15/100)}

= Rs 3x/20

Therefore, SP = (Listed Price) - (discount)

= Rs (x - 3x/20)

= Rs 17x/20.

But, the SP = Rs 15000

Therefore, 17x/20 = 15000

⇒ x = (15000 × 20/17)

⇒ x =17647.05

Hence, the marked price of the Fridge is Rs 17647.05

3.3. Discount Rate

It is the cost of the entire amount or quantity, which is typically less than its original worth. A total bill is typically sold at a discount, we can add. The discount is essentially the difference between the listed price and the selling price according to the profit and loss concept.

Selling price is the price at which the good or commodity has actually been sold, whereas marked price is the cost set by the seller in accordance with market norms. The buyer claims to have received a discount when the selling price is less than the marked price.

The discount is provided for the purpose of:

- raising sales,
- getting rid of obsolete stock,
- motivating distributors,
- rewarding potential buyers, etc.

Discount Rate Formula

1.Discount = Marked price – Selling price
2.Discount rate (DR) = p × r

Where p = principal amount

r = interest rate

3.3.1. Examples on Discount.

Question 1:

Sameer purchased a pant for Rs 4000. The discount offered for the shirt is 20%. Find how many Rs has he given to the cashier?

Solution:

Here, Principal amount p = 4000 dollars

Interest rate r = 20%

Discount rate, DR = p×r

DR = 4000×20% = 800

The discount amount for the dress is 800.

Discount rate, DR = 800.

Dress rate = Principal amount – Discount rate

= 8000 – 800 = 7200

Sameer has given Rs 7200 to the cashier.

Question 2:

Savita bought an old car for 80000 Rs in the year 2018. She sold the car by offering a 20% discount on the old car in the year 2022. Find how many Rs did she lose?

Solution:

Principal amount p = Rs 80000

Interest rate, r = 20%

Discount rate DR = p × r

Discount rate DR = 80000 × (20/100) in the year 2018

Discount rate =16000 Rs.

Thus, Selling price = Rs 80000 – Rs 16000 = Rs 64000

Hence, Savita lost Rs 16000

3.4. Annuities

An annuity is a steady flow of regular, equal payments made by one party to another for a certain amount of time in order to satisfy a financial commitment. The monetary value of the regular, equal payments made under an annuity is known as an annuity payment. An annuity with monthly payments for six months is depicted in the image below. Keep in mind that the payments are regular, continuous, equitable, and take place over a set period of time. If any one of these four requirements is not met, the financial transaction does not fulfil the definition of a singular annuity and must be solved using different methods and formulas.

3.4.1. Present Value of an Annuity

Given a specific rate of return, or discount rate, the present value of an annuity is the current value of the future payments from an annuity. The present value of the annuity decreases as the discount rate increases.

The amount of money that would be required today to fund a series of future annuity payments is referred to as the annuity's present value.

A sum of money received now is worth more than a similar sum at a later time due to the time value of money.

•If you want to know if you'll get more money from taking a lump amount now or an annuity spread out over several years, you can do a present value calculation.

Money received today has a higher value than the same amount received in the future since it can be invested while waiting due to the time value of money. By the same reasoning, $5,000 received today is more valuable than $5,000 divided into five $1,000/year instalments.

A discount rate is used to determine the value of money in the future. An interest rate or anticipated rate of return on other investments for the same period as the payments is referred to as the discount rate. The risk-free rate of return is the minimal discount rate applied in these computations. U.S. The return on Treasury bonds is frequently utilized for this purpose because they are typically thought of as the closest thing to a risk-free investment.

3.5. Fundamentals of Annuities

Before you are aware of a few of the crucial traits highlighted in the Unit introduction, it is impossible to compute payments effectively.

• Each of the given instances needed a regular payment of the same sum, e.g., your monthly mortgage payment of $872.41.

The payments were made at various intervals. While the mortgage and mattress purchases had the first payment due a month after the purchase, the car lease required the first payment to be made beforehand (at the beginning of the month) (at the end of the month).

• Both the interest rate and the payment frequency were variable. Unlike the vehicle lease, which had monthly payments and monthly interest, the mortgage required monthly payments with semi-annual interest.

3.6. Types of Annuities

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1. Timing of Payments. This trait is best demonstrated with an example. Let's say you borrow money today and make monthly payments on it. The amount of principal due would be quickly decreased and you would pay less interest during your first month if you made your first annuity payment on the day you took out the loan. Making your annuity payment at the start of a payment period is referred to as doing so, and this payment is referred to as a due. However, if a month goes by before you make your first loan payment, your original amount will accrue more interest than it would have otherwise.

   Making your payment at the conclusion of the payment interval is referred to as doing so, and this payment is referred to as an ordinary payment because it is the most typical type of annuity payment. You will pay a varying principle and interest amount depending on when you make your payment.

2. Frequency. The term "frequency of an annuity" refers to a comparison of the frequency of payments and the frequency of compounding. The number of annuity payments that would be made over the course of a year is known as a payment frequency. Remember from Unit 9 that the number of compounds each full year is the compounding frequency. A simple annuity is one in which the frequency of payments matches the frequency of compounding.

   You calculate principal and interest using streamlined formulae when interest is added to the account on a monthly basis and payments are likewise paid on a monthly basis. However, this is referred to as a general annuity if the frequency of payments and the frequency of compounding are dissimilar. The calculation of principal and interest requires the use of more complicated formulas if, for instance, payments are made monthly while interest is compounded semi-annually. This is because principal payments and interest charges are not made at the same time. Annuities versus Single Payments

Example of Present Value of an Annuity

$P = PMT\times [1-(1/(1+r)^n)]/r$

where:

P=Present value of an annuity stream

PMT = Dollar amount of each annuity payment

r = Interest rate (also known as discount rate)

n = Number of periods in which payments will be made

Assume a Mr X has the opportunity to receive an ordinary annuity that pays $40,000 per year for the next 20 years, with a 5% discount rate, or take a $550,000 lump-sum payment. Which is the better option? Using the above formula, the present value of the annuity is:

Present value=$\$40,000\times[1-(1/(1+0.05)^{20})]/0.05 = \$\ 498,488$

3.7. Summary

- ✔ Discount = Marked Price – Selling Price
- ✔ Discount Percentage Formula = Marked Price × Discount Rate
- ✔ Discount rate (DR) = P × r
- ✔ $P = PMT\times [1-(1/(1+r)^n)]/r$

3.8. Self-Assessment questions

1. What is a discount?
2. Write all formulae of discount?
3. What is the discount rate?
4. A dealer purchased a Fridge for Rs 20000. He offered a discount of 20% on its listed price and still gained 5%. Find the listed price of the Fridge.
5. Pranita bought the vehicle for 90000 Rs in the year 2015. She sold the car by offering a 10% discount on the old car in the year 2018. Find how many Rs did she lose?
6. What is annuity?
7. Explain present value of annuity

8. Write fundamental of annuity
9. Explain types of annuities
10. Write the difference between discount and annuity.

3.9. Reference

- ✔ https://byjus.com/maths/discount-rate/
- ✔ https://www.vedantu.com/formula/discount-formula

Unit 4

Solving Simultaneous Equations and Quadratic Equations

Learning Objectives

After studying the unit, students will be able to:

- Concept of linear equation
- Solving simultaneous equation by elimination method
- Solving simultaneous equation by substitution method
- Study of quadratic equation
- Roots of quadratic equation by factorisation
- Roots of quadratic equation by formula

Structure

4.1. Introduction

4.2. Simultaneous Equations

4.3. Elimination Method

4.4. Substitution Method

4.5. Quadratic equations

4.6. Factorization of Quadratic Equation

4.7. Quadratic Formula

4.8. Summary

4.9. Self-assessment questions

4.10. Reference

4.1. Introduction

- Let's review the definition of equations in mathematics and the many forms of equations before moving on to simultaneous equations. An equation in mathematics is a statement that calls for the equality of two variables. Two expressions are placed one on either side of the equal symbol (=) to form an equation. There are two or more variables in it. In essence, the LHS value must match the RHS value. If you change the values of the variables in an equation, you should be able to demonstrate equality. In mathematics, there are various equation types, including:
- Linear Equation
- Quadratic Equation
- Polynomial Equations

4.2. Simultaneous Equations

An equation that comprises many quantities that are connected by multiple equations is known as a simultaneous equation. There are a couple independent equations in it. The system of equations, commonly referred to as the simultaneous equations, is made up of a finite number of equations for which a common solution is sought. Finding the values of the variables in the equations is necessary in order to solve them.

You will discover here how to solve simultaneous linear equations using examples.

The simultaneous linear equations' general form is as follows:

$ax + by = c$

$dx + ey = f$

4.2.1. Methods for Solving Simultaneous Equations

- There are several ways to solve the simultaneous linear equations. The simultaneous equations can be solved using one of three methods: substitution, elimination, or the augmented matrix method. The two easiest of these three approaches will successfully resolve the simultaneous equations and yield precise solutions. Here, we'll talk about these two crucial techniques, namely:
  - Elimination Method
  - Substitution Method

4.3. Elimination Method

To comprehend how to solve simultaneous equations using the elimination approach, follow the procedures in the example that has been solved below.

Example: Solve the following simultaneous equations using the elimination method.

3a + 3b = 15

4a – 3b = 6

Solution:

The two given equations are

3a + 3b = 15…….(1)

4a – 3b = 6……….(2)

Step 1: The coefficient of variable 'b' is equal and has the opposite sign to the other equation. Add equations 1 and 2 to eliminate the variable 'b'.

Step 2: The like terms will be added.

(3a+4a) +(3b – 3b) = 15 + 6

7a = 21

Step 3: Bring the coefficient of a to the R.H.S of the equation

a = 21/ 7

Step 4: Dividing the R.H. S of the equation, we get a = 3

Step 5: Now, substitute the value a=3 in the equation (1), it becomes

3(3) + 3b = 15,

9 + 3b = 12

3b = 12-9

3b =3

b = 3/3 = 1

Step 6: Hence, the solution for the given simultaneous equations is a = 3 and b = 1.

4.4. Substitution Method

To help you better grasp how to solve simultaneous linear equations using the substitution approach, we've included a solved example with steps below.

Example: Solve the following simultaneous equations using the substitution method.

b= a + 3

a + b = 7.

Solution:

The two given equations are

b = a + 3 ————(1)

a + b = 7 —————(2)

We will solve it step-wise:

Step 1: Substitute the value of b into the second equation. We will get,

a + (a + 3) = 7

Step 2: Solve for a

a +a + 3 = 7

2a + 3 = 7

2a = 7 – 3

a = 4/2 = 2

Step 3: Substitute this value of a in equation 1

b = a + 3

b = 2 + 3

b = 5

step 4: Hence, the solution for the given simultaneous equations is: a = 2 and b = 5

- **Addition/subtraction method**

This method is also known as the elimination method.

To use the addition/subtraction method, do the following:

1. Multiply one or both equations by some number(s) to make the number in front of one of the letters (unknowns) the same or exactly the opposite in each equation.
2. Add or subtract the two equations to eliminate one letter.
3. Solve for the remaining unknown.
4. Solve for the other unknown by inserting the value of the unknown found in one of the original equations.

Example 1

Solve for *x* and *y*.

$$x + y = 7$$

$$x - y = 3$$

Adding the equations eliminates the *y*-terms.

$$\begin{array}{r} x + y = \ \ 7 \\ x - y = \ \ 3 \\ \hline 2x \qquad = 10 \end{array}$$

$$\frac{2x}{2} = \frac{10}{2}$$

$$x = 5$$

Now inserting 5 for *x* in the first equation gives the following:

$$\begin{array}{rr} 5+y= & 7 \\ -5 & -5 \\ \hline y= & 2 \end{array}$$

Answer: *x* = 5, *y* = 2

Example 2

Solve for *x* and *y*.

$$3x+3y=24$$
$$2x+y=13$$

First multiply the bottom equation by 3. Now the *y* is preceded by a 3 in each equation.

$$\begin{array}{rr} 3x+3y=24 & 3x+3y=24 \\ 3(2x)+3(y)=3(13) & 6x+3y=39 \end{array}$$

The equations can be subtracted, eliminating the *y* terms.

$$\begin{array}{rrr} 3x+ & 3y & = 24 \\ -6x+ & (-3y) & =-39 \\ \hline -3x & & =-15 \end{array}$$

$$\frac{-3x}{-3}=\frac{-15}{-3}$$

$$x=5$$

Insert *x* = 5 in one of the original equations to solve for *y*.

$$\begin{array}{rr} 2x+y= & 13 \\ 2(5)+y= & 13 \\ 10+y= & 13 \\ -10 & -10 \\ \hline y= & 3 \end{array}$$

Answer: *x* = 5, *y* = 3

4.5. Quadratic equations

Quadratic equations are the polynomial equations of degree 2 in one variable of type $ax^2 + bx + c = 0$ where a, b, c, ∈ R and a ≠ 0.

A Quadratic equation is a second-degree equation of the form $ax^2 + bx + c = 0$. Here a, b, are the coefficients, c is the constant term, and x is the variable

Having the form $ax^2 + bx + c = 0$, quadratic equations are second-degree algebraic expressions. From the term "quad," which means square, comes the word "quadratic." A quadratic equation is a "equation of degree 2," to put it another way. A quadratic equation is employed in numerous situations. Did you know that a quadratic equation can accurately predict a rocket's trajectory after launch? In addition, there are many uses for quadratic equations in physics, engineering, astronomy, etc.

Roots of a Quadratic Equation

When a quadratic equation is solved, the two values of x that result are known as the roots of the equation. The zeros in the equation are another name for these quadratic equation roots. For instance, x = -1 and x = 4 are the roots of the equation $x^2 - 3x - 4 = 0$ since they each fulfil it.

- At x = -1, $(-1)^2 - 3(-1) - 4 = 1 + 3 - 4 = 0$
- At x = 4, $(4)^2 - 3(4) - 4 = 16 - 12 - 4 = 0$

Methods to Solve Quadratic Equations

You can solve a quadratic equation to get the two roots of the equation or two values of x.

4.6. Factorization of Quadratic Equation

Follows a set of actions. We must first divide the middle word into two terms so that the sum of the terms equals the constant term in order to obtain a generic form of the quadratic equation $ax^2 + bx + c = 0$. To eventually obtain the necessary factors, we can take the common terms from the term that is available. The following general representation of the quadratic equation can be used to comprehend factorization.

Example 1

- $x^2 - 5x + 6 = 0$
- $x^2 - 2x - 3x + 6 = 0$
- $x(x - 2) + 3(x - 2) = 0$
- $(x - 2)(x - 3) = 0$

Thus, the two obtained factors of the quadratic equation are (x - 2) and (x - 3).

To find its roots, just set each factor to zero and solve for x. i.e., x - 2 = 0 and x - 3 = 0 which gives x = 2 and x = 3.

Thus, x = 2 and x = 3 are the roots of $x^2 - 5x + 6 = 0$

Example 2

- $x^2 + 7x + 12 = 0$
- $x^2 + 4x + 3x + 12 = 0$
- $x (x + 4) + 3(x + 4) = 0$
- $(x + 4) (x + 3) = 0$

Thus, the two obtained factors of the quadratic equation are (x +4) and (x + 3).

To find its roots, just set each factor to zero and solve for x. i.e., x + 4 = 0 and x + 3 = 0 which gives x = - 4 and x = - 3.

- Thus, x = - 4 and x = 3 are the roots of $x^2 + 7x + 12 = 0$

Example 3

- $x^2 - 9x - 36 = 0$
- $x^2 - 12x + 3x - 36 = 0$
- x (x - 12) + 3(x - 12) = 0
- (x - 12) (x + 3) = 0

Thus, the two obtained factors of the quadratic equation are (x - 12) and (x + 3).

To find its roots, just set each factor to zero and solve for x. i.e., x - 12 = 0 and x + 3 = 0 which gives x = 12 and x = - 3.

- Thus, x = 12 and x = - 3 are the roots of $x^2 - 9x - 36 = 0$

4.7. Quadratic Formula

The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1 : Find the roots of the equation $x^2 - 2x - 5 = 0$ using the quadratic formula.

Solution:

a = 1, b = -2, and c = -5.

$x = [-b \pm \sqrt{(b^2 - 4ac)}]/2a$

$= [-(-2) \pm \sqrt{((-2)^2 - 4(1)(-5))}]/2(1)$

$= [2 \pm \sqrt{24}] / 2$

$= [2 \pm 2\sqrt{6}] / 2$

$= (2 + 2\sqrt{6})/2$ or $(2 - 2\sqrt{6})/2$

Example 2: Find the roots of the equation $x^2 - 8x - 20 = 0$ using the quadratic formula.

Solution:

a = 1, b = -8, and c = -20.

$x = [-b \pm \sqrt{(b^2 - 4ac)}]/2a$

$= [-(-8) \pm \sqrt{((-8)^2 - 4(1)(-20))}]/2(1)$

$= [8 \pm \sqrt{144}] / 2$

$= [8 \pm 12] / 2$

$= (8 + 12)/2$ or $(8 - 12)/2$

$= 10$ or -2

4.8. Summary

- The simultaneous linear equations' general form is ax +by =c
- Methods of simultaneous linear equations are elimination and substitution methods.
- We can solve quadratic equation by factorisation method
- Quadratic formula $x = [-b \pm \sqrt{(b^2 - 4ac)}]/2a$

4.9. Self-assessment question

1. Solve: 7x + 5y = 12 and 4x + 5y = 10
2. Solve for a and b:

   10a – 8b = 6

   10a – 9b = -2
3. What are methods of simultaneous equations?
4. Write down the steps of the elimination method.
5. Solve: 3x + 4y = 24 and 4x + 3y = 11
6. Define a quadratic equation.
7. Solve by factorisation method $x^2 - x - 56 = 0$
8. Solve by factorisation method $x^2 - 7x + 12 = 0$
9. Solve the quadratic equation by formula $x^2 - x - 30 = 0$
10. Solve the quadratic equation by formula $x^2 - 3x - 4 = 0$

4.10. Reference

- https://byjus.com/maths/simultaneous-equations/
- https://www.cuemath.com/algebra/quadratic-equations/

Unit 5

Progression (AP, GP, HP)

Learning objectives

After studying the unit, students will be able to:

- Idea of AP
- Concept of GP
- Concept of HP
- Various examples of AP, GP, HP

Structure

5.1. Introduction

A progression is nothing more than a pattern of numbers, commonly referred to as a series. For instance, the sequence 4, 7, 10, 13……. is a progression because it follows a pattern in which each successive number is created by adding 3 to the one before it. But not every advancement must follow the same pattern.

Its pattern is determined by the kind of advancement. Let's examine the many progression types, illustrations, and rules. What Constitutes a Progression, exactly?

A progression is a collection of figures (or things) that follow a specific pattern. A sequence is another name for a progression. Every term in a progression is created by applying a certain rule to the word before it. To put it another way, every term in a progression is defined by the general term (or) nth term, which is indicated by the letter an. Types of Progressions

There are mainly 3 types of progressions in math. They are:

- Arithmetic Progression (AP)
- Geometric Progression (GP)
- Harmonic Progression (HP)

5.2. Arithmetic Progressions (AP)

When there is no change in the difference between two consecutive numbers, a series of numbers is said to be in arithmetic progression.

It also indicates that the previous number in the series can be added to or subtracted from to get the subsequent number. Consequently, the common difference is the name given to this constant number (d).

For example, 5, 10, 15, 20, 25 is an AP as the difference between two consecutive terms is 10 which is fixed.

Suppose if, ‘a’ is the first term and ‘d’ is a common difference, then,

Formula to find nth term of arithmetic progression is

$$t_n = a + (n-1)d$$

Sum of n terms of AP

$$s_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Example 1: What is the 8^{th} term of the sequence 1, 3, 9, 27, ...?

Solution:

Observe that 3/1 = 9/3 = 27/9 = ... = 3.

Hence, the given sequence is a geometric progression (GP) where the first term is a = 1 and the common ratio is r = 3.

The formula for n^{th} term of a GP is,

$t_n = ar^{n - 1}$

For 8^{th} term, substitute n = 8.

$t_8 = 1(3)^{8 - 1} = 3^7$.

Answer: The tenth term = 3^7.

Example 2: Which term of the AP 4, 9, 14, 19, ... is 74?

Solution:

In the given arithmetic progression,

First term, a = 4

Common difference, d = 9 - 4 = 14 - 9 = 19 - 14 = ... = 5.

Let its n^{th} term = 7.

a + (n - 1) d = 75

4 + (n - 1) 5 = 74

4 + 5n - 5 = 74

5n -1 = 74

5n = 75

n = 15

Answer: 74 is the 15^{th} term of the given AP.

Example 3: The 6^{th} term and the 11^{th} terms of a harmonic progression are 1/10 and 1/20 respectively. Find the common difference of the associated AP.

Solution:

From the given information, the 6^{th} and 11^{th} terms of the corresponding AP are 1/10 and 1/18 respectively. Let 'a' be the first term and 'd' be the common difference of the associated AP.

Then using the n^{th} term formula of AP,

a + 5d = 10 ... (1)

a + 10d = 20 ... (2)

Subtracting (1) from (2),

5d = 20 - 10

5d = 10

d = 2.

Answer: The common difference of AP is 2.

5.3. Geometric Progression

If any two succeeding terms' ratios are consistently similar, the sequence is known as a geometric progression.

It can alternatively be described as a series in which the subsequent number can be generated by multiplying the prior number in the series by a constant. Consequently, the common ratio is the name given to this fixed word (r).

Quantities are termed as GP if they tend to increase and decrease by a common factor.

Suppose, if 'a' is the first term and 'r' be the common ratio, then

- Formula for nth term of GP and sum of n terms of GP

$$t_n = ar^{n-1}$$

$$s_n = \frac{a\left(r^n - 1\right)}{r - 1}; r > 1$$

$$s_n = \frac{a\left(1 - r^n\right)}{1 - r}; r < 1$$

5.3.1. Examples of Geometric Progression

Question 1: If the first term is 8 and the common ratio of a GP is 3, then write the first four terms of GP.

Solution: Given,

First term, a = 8

Common ratio, r = 3

We know the general form of GP for first five terms is given by:

a, ar, ar^2, ar^3, ar^4

a = 8

ar = 8 × 3 = 24

$ar^2 = 8 \times 3^2 = 8 \times 9 = 72$

$ar^3 = 8 \times 3^3 = 216$

Therefore, the first four terms of GP with 10 as the first term and 3 as the common ratio are:

8, 24, 72, 216

Question 2: Find the sum of five terms of GP: 10, 40, 160, 640……, using formula.

Solution: Given GP is 10, 40, 160, 640

First term, a = 10

Common ratio, r = 40/10 = 4 > 1

Number of terms, n = 5

Sum of GP is given by;

$S_n = a[(r^n – 1)/(r – 1)]$

$S_5 = 10[(4^5 – 1)/(4 – 1)]$

= 10[(1024 – 1)/3]

= 10[1023/3]

= 10 × 341

= 3410

Question 3: If 3, 6, 12…., is the GP, then find its 10th term.

Solution: The nth term of GP is given by:

3, 6, 12….,

Here, a = 3 and r = 6/3 = 2

$t_n = ar^{n-1}$

Therefore,

$t_{10} = 3 \times 2^{10 – 1}$

$= 3 \times 2^9$

= 1536

5.4. Harmonic Progression

Taking the reciprocals of the arithmetic progression that does not contain zero yields a sequence of real numbers known as a harmonic progression (HP). Any phrase in a harmonic progression is regarded as the harmonic mean of its two neighbours. An arithmetic progression, for instance, would be the sequence a, b, c, d, etc.; a harmonic progression would be expressed as 1/a, 1/b, 1/c, 1/d, …

Harmonic Progression Formula

We need to identify the corresponding arithmetic progression sum in order to solve the harmonic progression difficulties. It denotes that the reciprocal of the nth term of the related A.P. is equivalent to the

nth term of the harmonic progression. As a result, the following is the formula to determine the nth term in the harmonic progression series:

The nth term of the Harmonic Progression (H.P) = 1/ [a+(n-1)d]

Where

"a" is the first term of A.P

"d" is the common difference

"n" is the number of terms in A.P

The above formula can also be written as:

The nth term of H.P = 1/ (nth term of the corresponding A.P)

5.4.1. Harmonic Progression Examples

Example 1:

Determine the 5th and 9th term of the harmonic progression 1/4, 1/7, 1/10, …

Solution:

Given:

H.P = 1/4, 1/7, 1/10, …

Now, let us take the arithmetic progression from the given H.P

A.P = 4,7,10 ….

Here, $T_2 - T_1 = T_3 - T_2 = 3 = d$

So, in order to find the 5th term of an A. P, use the formula,

The nth term of an A.P = a + (n-1) d

Here, a = 4, d= 3

Now, we have to find the 5th term.

So, take n=5

Now put the values in the formula.

5th term of an A.P = 4 +(5-1) 3

= 4 +12

= 16

Similarly,

9th term of an A.P = 4 +(9-1) 3

= 4 +24

= 28

Since H.P is the reciprocal of an A.P, we can write the values as:

5th term of an H.P = 1/5th term of an A.P = 1/16

9th term of an H.P = 1/9th term of an A.P = 1/28

Example 2:

Compute the 15th term of HP if the 5th and 10th term of HP are 1/10 and 1/22, respectively.

Solution:

The H.P is written in terms of A.P are given below:

5th term of A.P = a+4d = 10 —- (!)

10th term of A.P = a+10d = 22 …… (2)

By solving these two equations, we get

a =2, and d = 2

To find 15th term, we can write the expression in the form,

a+14d = 2 +28 = 30

Thus, the 15th term of an H.P = 1/15th term of an A.P = 1/30

Therefore, the 15th term of the H.P is 1/30

5.5. Summary

- In an arithmetic progression, each successive term is obtained by adding the common difference to its preceding term.
- In a geometric progression, each successive term is obtained by multiplying the common ratio to its preceding term.
- The reciprocal of terms in harmonic progression form an arithmetic progression.

- AP $t_n = a + (n-1)d$ $s_n = \frac{n}{2}[2a + (n-1)d]$
- GP

$$t_n = ar^{n-1}$$

$$s_n = \frac{a(r^n - 1)}{r - 1}; r > 1$$

$$s_n = \frac{a(1 - r^n)}{1 - r}; r < 1$$

5.6. Self-assessment questions

1. What is the 12th term of the sequence 4, -8, 16, -64.
2. Check whether the given sequence is GP.
   27, 9, 3, …
3. Write the first five terms of a GP whose first term is 3 and the common ratio is 2.
4. Example 2: Compute the 100th term of HP if the 10th and 20th terms of HP are 20 and 40 respectively.

5. Find the 1st term of the AP whose 7th and 11th terms are respectively 37 and 57.
6. How many natural numbers between 200 to 500 are multiples of 3?
7. The 8th term of a GP is 16 times the 4th term. What will be the first term when its sixth term is 64.
8. Explain AP and GP
9. What is HP is it related with AP
10. Find the number of terms in the series 1/8, 1/2, 2…….8192.

5.7. Reference

- ✔ https://www.geeksforgeeks.org/progressions-apgp-hp-set-2/
- ✔ https://unacademy.com/content/jee/study-material/mathematics/relation-between-apgp-and-hp/

Unit 6

Set

Learning Outcomes:

After studying the unit, students will be able to:

- Representation and notation of set
- Various set operations
- Different types of set
- Some special sets
- Properties and laws of sets

Structure

6.1 Introduction

Set notation describes the many symbols used while navigating between and within sets. The curly brackets symbol is the most basic way to express a set's components. A = a, b, c, and d is an illustration of a set. Here, a capital letter designates the set, while little letters stand in for its constituent parts. Set representation and set operations are two main categories for set notation. The notations for representing sets are universal set, null set, subset, belongs to, and complement of a set (A'). Additionally, U, union, intersection, difference, and delta are the set notations for operations across sets. Using examples and FAQs, let's learn more about the various set notations.

We all understand that items that can be counted, such as numbers, are those that fall into a number series and may be listed sequentially into a reference with the numbers. These objects can be grouped together if they share similar traits, share similar goods, or even offer similar services. Here, we need to discuss the word "set notation," but before we do, it will help if we define "sets" first. By doing this, the notion of "set notation" will be simpler to comprehend. So, in essence, what is a set? Sets are nothing more than, as their name implies, a collection of related countable and identifiable items that provide statistical information that helps us comprehend a set's characteristics.

To distinguish it from other types of sets, a set is defined as the integration of several objects with comparable qualities into a single correctly bonded pair of brackets. In general, sets can be described using pairs of curly brackets, commonly known as the second bracket, or "{ }".

We now have a clear framework for discussing the idea of set notation. Set notation, then, typically refers to the method of expressing the quantity of sets and their objects between brackets. In plain English, notation is the usage of symbols. Understanding the many symbols that may be used to indicate various types of sets as well as the nature of the set or the items included inside it enables us to better grasp set notation. The items that fall into the category of being defined based on their nature are commonly referred to as elements. It follows that the phrases "objects," "elements," and "members" are comparable terminology used to define the numbers contained within a pair of sets.

So, in essence, what is a set? Sets are nothing more than, as their name implies, a collection of related countable and identifiable items that provide statistical information that helps us comprehend a set's characteristics.

To distinguish it from other types of sets, a set is defined as the integration of several objects with comparable qualities into a single correctly bonded pair of brackets. In general, sets can be described using pairs of curly brackets, commonly known as the second bracket, or "{}".

The fundamental symbols used to express the multiple representations throughout set operations are called set notations. Any work done both within and between the sets is indicated using set notation. With the exception of the number elements, all symbols may be taken to represent notations for sets. The Curly brackets, which are used to surround and symbolise the set's components, are the most basic form of set notation. Using parentheses or floral brackets, the components of a set are written ( ).

A set's components are listed and separated by commas. The English alphabet's five vowels are found in set A, which is represented as A = a, e, I o, u. The elements of the sets are identified by tiny letters, whereas the sets themselves are identified by capital letters. Additional sets and actions are represented using set notation. Additionally, only with the aid of set notation is it possible to represent the numerous relations and functions across sets.

The set notations may be broadly divided into two categories: set representations and set operations. The set notation for set representation and the set construction for set operations will now be thoroughly examined.

6.2. Set Notation for Set Representation

The universal set, empty set, and the complement of a set are three of the typical sets that are commonly represented using set notation. To further connect the elements or the set itself to another set, we utilise special symbols for subset and belongs to.

6.2.1.µ - Universal Set: This contains every element of every set that is taken into account while performing set operations. The universal set represents all the English alphabets if there is a set A of consonants and a set B of consonants. As a result, a universal set is a set that contains every element of every set being considered. We all have obviously come across such numbers that whenever or wherever we go, will always give us a fixed value for fixed pairs of characteristics and properties. Similarly, it will be clear for us to understand the term universal concept, if we go through the first statement. It simply says that, all the objects or elements or members of different small sets will come under one common set, and that one common set which is common for all the other sets will be considered as the universal set. The universal set is simply denoted with the letter, “U” or “mu”.

Yeah of course, it will be more helpful if we go through an example. If one asks you, just name some beverages that are famous worldwide. Then it is obvious that one will answer it as Coca Cola, Pepsi, Thumbs Up, Sprite, etc. Since, we all know that, all these countable names of beverages which are famously known, come under one feature and that is the set of beverages. This can be written as, Beverages = { Coca Cola, Pepsi, Thumbs Up, Sprite, etc}.

This makes it clear that beverages, here in this case, are the universal set. For all other sets of soft drinks, cold coffees, shakes, fruit juice, all these come under one set, that is beverages, thus, making the beverages as a universal set, in this case.

6.2.2.Ø - Null Set: A set is said to be a null set if it contains no elements. It is also known as an empty set and is denoted by the symbol Ø = { }. The term null means nothing which means empty. The null sets simply implies all about the Empty sets. It will be more clear to understand if we take an example. If we consider, a set named E which implies that E is the set of cube numbers, where there is one condition and that condition is that all these cube numbers must have to be odd. And if we write it in the set builder format then, we get,

E = {y: cube of y = 8, is odd}

But when we solve this set, then we get that the result turns out to be as there are no such numbers possible to be formed. Thus, set E here becomes the empty or null set. This means that, there are no elements present in any of the empty set or null set.

Null set is simply represented with empty curly brackets like, { }, or as the symbol of "phi"

A' - the components of the universal set, with the exception of the components of set A, are considered to be a set's complement. The complement of a set A is A' = μ - A. If A = 2, 3, 4, 5 and μ = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 for the set, then A' = - A = 1, 6, 7, 8, 9, 10.

6.2.3. ⊂- Subset: A set created by selecting a few members from a larger set is represented by the subset symbol. If a new set B = b, c, d is created by plucking a few members from a set A = a, b, c, d, e, then we say that B is a subset of A, and this is symbolised as B ⊂ A. B is referred to as the correct subset of A since it has less members than set A. Additionally, B would be indicated as B A if it included exactly the same components as set A. Subset means here, there will be two or more than two sets, where one needs to know to differentiate between the small sets and the larger sets, so that it becomes easier to work with subsets. If one set contains some of the elements same as that of the other sets, then depending upon the number of elements that one set holds, will give us that one set is the subset of the other set.

For example, Let us consider, two sets, set A and set B, where,

A = {set of all irrational numbers}

B = {set of all real numbers}

Here, we can say that set A is a subset of set B. This is possible because real numbers are combinations of both irrational and rational numbers, which means elements of set A are also in the elements of set B including the rational numbers too. So, set A is the subset of set B.

6.2.4. ∈ - Belongs to: If a certain element is stated to belong to a set A, the symbol "∈ "is used. The element an is referred to as belonging to set A if the set A = a, b, and c by the notation a ∈ A. Additionally, we designate an element d as d ∈ set A if it does not belong to the set A. It does not belong to the set if it has ∈ the sign.

6.3. Set Notation for Set Operations

Set notation helps in conducting various operations across sets. The operations of union, intersection, difference, delta, across two or more sets are denoted by the various set notations, using symbols.

6.3.1. U - Union of Sets: This operation of union of sets combines all the elements of the two sets and presents it as a single set. Here the common elements of the sets are written only once in the final union set. For two sets A = {a, b, c, d}, and B = {c, d, e}, we have A U B = { a, b, c, d, e}. The elements in the union set have more elements than in the individual sets.

Union of sets is one of the operations that we will be going through in the further discussions. So, what does union in general mean? Union, in simple words means to combine or accumulate, altogether.

Thus, union of sets simply means to combine or one can say to integrate different numbers of sets, which either might be having same features or in some cases, may not. Here, the elements or objects or one can say as members, are combined together in order to come to a result by forming it as one single set, so as to understand the similarity between the two sets.

Union of sets are symbolised as,

This concept will be more clear as water to all, if we go through an example.

So, let us consider, two sets, and let us name them as, Set E and the other as Set O, which can be written as,

E = {2,4,6,8,10,12,14}

O = {1,3,5,7,9,11,13,15}

Therefore, E O = { 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15}

It means, when any two of the sets are combined together, then the resultant set that we get is termed as the union of sets.

Union of sets are sometimes called as or operators.

6.3.2. ∩ - Intersection of Sets: The operation of intersection of sets takes the common elements of the two sets to form a new set. For the sets A = {a, b, c, d, e, f}, B = {e, f, g, h, i}, we have A ∩ B = {e, f}. Here we have taken the common elements and the number of elements in the intersection set is lesser than the elements in the individual sets. The term intersection of sets means that when two or more sets are being combined then we only write those objects or elements that are common in both those old sets. This will be more clear, if one says, intersection in common language means "and", means one can only write those elements which are common in one set "and" in the other set. Intersection of sets is symbolised as,

- For example, if we consider two sets, set G and set H,
- G = { a,b,c,d,e }
- H = { b,d,f,h }
- Therefore, G H = { b,d }
- This means that the intersection of Set G and Set H, has those elements, which are common in both Set G and Set H.

6.3.3. (A - B or B-A) Difference: The difference of the two sets uses the same symbol of subtraction. Here the elements remaining after removing the common elements from the given set give the difference between the two sets. For the sets A = {a, b, c, d, e}, B = {b, c, d}, we have the difference of sets, between the sets A - B = A - (A ∩ B) = {a, b, c, d, e} - {b, c, d} = {a. e}.The term difference, itself as the name says, means that, it means the subtracted value that we get after making the subtraction between the two sets, in general when we subtract the smaller set from the bigger set. The symbol that is used to represent the difference operation of sets is, "-".

- For example, if we consider two sets Set 1 and Set 2,
- 1 = { p,q,r,s,t,u,v}
- 2 = { q,r,t,u}
- Therefore, 1-2 = { p,s,v }
- So, in the above example, we can see that, when the smaller set, set2 got subtracted from the bigger set, set 1, the resultant output that one can see is that, the elements that were in set 2 got removed from the set 1 and then we got a new set, as (1-2).

6.3.4. A' - Complement: For a set A, its complement is written as A' or A^c. The complement of a set is the set obtained after removing the given set elements from the universal set and taking the remaining elements. The formula for the complement of a set is A' = μ - A. Complement operation is something quite similar as we got in the above sections about one set and about the universal set.

- Complement of a set simply means that this set contains all the other elements which are not present in the previous set, which holds the same name. This line will be more clear to you all, if we go through an example. So, let us first consider a set, let us name it as set A., which contains some numbers as its elements.
- A = { 2,4,6,8,10,12,14}
- We all know that the numbers or the elements which are in the set A, come from the Universal set of natural numbers.
- N = { 1,2,3,4,5,6,7,8,9,10,11,12,...infinity }
- And now, if one wants to find the complement of set A, which is represented as A', will be written as,
- A' = { 1,3,5,7,9,11,13 }
- It means the complement of set A contains all those numbers which are not in the set A, where both set A and complement of set A belong to the universal set of N.

6.3.5.Δ - Delta: The notation of delta between two sets gives the elements remaining after removing the common elements from the union of the two sets. For two sets A, and B we have A Δ B = (A U B) - (A ∩ B), Or A Δ B = (A - B) U (B - A). Also the A Δ B takes the elements of only set A and only set B.This is a term which is quite different from others, as well as can be more complex too. So, it is mandatory for one to understand it very well. And to understand it well, one must go through the other operations properly, so that they don't get any doubt while solving sums regarding this operation.

- Basically, delta operation is all about excluding those elements or objects or members from those sets that one gets as the common objects, that too after combining the other elements of the original two sets. It means when two sets are combined, mathematically known as union of sets, then there are some resultant common elements, which are further removed from that particular set, so as to give us the final set that represents the delta operation.
- To make this concept more clear and more practically understandable, we need to take an example. So, for instance, if we take two sets, Set A and Set B,
- A = { 1,2,3,4,5,6,7,8,9,10 }
- B = { 1,4,9,16,25 }
- If we do, A union B, we get,
- A union B = { 1,2,3,4,5,6,7,8,9,10,16,25 }
- Then, if we do A intersection B, we get,
- A intersection B = { 1,4,9 }

- Now, we have,
- A delta B = (A union B) - (A intersection B) = { 2,3,5,6,7,8,10,16,25 }
- Here, we got the value of A delta B, first after doing the union between the sets A and B, followed by intersection of sets A and B, secondly, after subtraction the interaction of sets A and B from the union of sets A and B.
- There is another way of doing it, in short, we can do it directly too.
- First, let us do A difference B, we get,
- A-B = { 2,3,5,6,7,8,,10 }
- Then, we need to do, B difference A, we get,
- B-A = { 16,25 }
- Now, we need to do the union of both the differences that we have done, such that, we get,
- A delta B = (A-B) union (B-A) = { 2,3,5,6,7,8,10,16,25 }

6.4. Different Types of Sets

Set types are categorised based on how many pieces they include. Elements of the same type are gathered in sets. a collection of prime numbers, natural numbers, etc. are a few examples. There are many different kinds of sets, including equal and unequal sets, empty sets, finite and infinite sets, and unit sets. Let's take a closer look at the various types of sets.

6.4.1. Singleton Sets or Unit Sets

A set that has only one element is called a singleton set. It is also known as a unit set because it has only one element. Example, Set A = { k | k is an integer between 5 and 7} which is A = {6}.

6.4.2. Finite Sets

As the name implies, a set with a finite or exact countable number of elements is called a finite set. If the set is non-empty, it is called a non-empty finite set. Some examples of finite sets are: For example, Set B = {k | k is an even number less than 20}, which is B = {2,4,6,8,10,12,14,16,18}. Let us consider one more illustration, Set A = {x : x is a day in a week}; Set A will have 7 elements.

6.4.3. Infinite Sets

A set with an infinite number of elements is called an infinite set. In other words, if a given set is not finite, then it will be an infinite set. For example, A = {x: x is a real number}; there are infinite real numbers. Hence, here A is an infinite set. Let us consider one more example, Set B = {z: z is the coordinate of a point on a straight line}; there are infinite points on a straight line. So, here B is an example of an infinite set. Another example could be Set C = {Multiples of 3}. Here we can have infinite multiples of 3.

6.4.4. Empty or Null Sets

A set that does not contain any element is called an empty set or a null set. An empty set is denoted using the symbol '∅'. It is read as 'phi'. Example: Set X = {}.

6.4.5. Equal Sets

If two sets have the same elements in them, then they are called equal sets. Example: A = {1,3,2} and B = {1,2,3}. Here, set A and set B are equal sets.
This can be represented as A = B.

6.4.6. Unequal Sets

If two sets have at least one element that is different, then they are unequal sets. Example: X = {4, 5, 6} and Y = {2,3,4}. Here, set X and set Y are unequal sets. This can be represented as $X \neq Y$.

6.4.7. Equivalent Sets

Two sets are said to be equivalent sets when they have the same number of elements, though the elements are different. Example: A = {7, 8, 9, 10} and B = {a,b,c,d}. Here, set A and set B are equivalent sets since n(A) = n(B)

6.4.8. Overlapping Sets

Two sets are said to be overlapping if at least one element from set A is present in set B. Example: A = {4,5,6} B = {4,9,10}. Here, element 4 is present in set A as well as in set B. Therefore, A and B are overlapping sets.

6.4.9. Disjoint Sets

Two sets are disjoint sets if there are no common elements in both sets. Example: A = {1,2,3,4} B = {7,8,9,10}. Here, set A and set B are disjoint sets.

6.5. Subset and Superset

For two sets A and B, if every element in set A is present in set B, then set A is a subset of set B(A ⊆ B) and B is the superset of set A(B ⊇ A).

Example: A = {1,2,3} B = {1,2,3,4,5,6}

A ⊆ B, since all the elements in set A are present in set B.

B ⊇ A denotes that set B is the superset of set A.

6.5.1. Universal Set

A universal set is the collection of all the elements in regard to a particular subject. The set notation used to represent a universal set is the letter 'U'. Example: Let U = {The list of all road transport vehicles}. Here, a set of cars is a subset for this universal set, the set of cycles, trains are all subsets of this universal set.

6.5.2. Power Sets

Power set is the set of all subsets that a set could contain. Example: Set A = {1,2,3}. Power set of A is = {{∅}, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}}.

6.6. Set Operations

Similar to basic operations on integers, set operations are a notion. Mathematical sets deal with a finite number of items, such as letters, numbers, or any other real-world objects. There are instances when it becomes necessary to determine the connection between two or more sets. The idea of set operations is introduced here.

Set union, set intersection, set complement, and set difference are the four primary set operations. The different set operations, set representation notations, set operations, and set usage in practical applications will all be covered here.

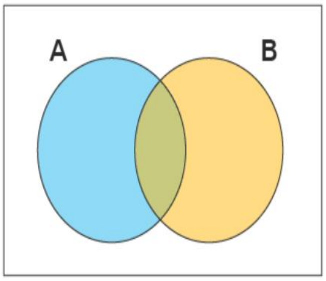
- **What are Set Operations?**

A group of things is referred to as a set. An "Element" is the name for each item that makes up a set. Three ways can be used to represent a set. They are set builder notation, roster form, and statement form. Set operations are actions taken on two or more sets to establish a connection between them. The following are the four primary categories of set operations.

- Union of sets
- Intersection of sets

- Complement of a set
- Difference between sets/Relative Complement

Let's review the Venn diagram notion before talking about the different set operations because it's crucial to knowing how sets work. A Venn diagram is a logical illustration that demonstrates potential connections between several finite sets. The following is a representation of the Venn diagram.



- **Basic Set of Operations**

Now that we know the concept of a set and Venn diagram, let us discuss each set operation one by one in detail. The various set operations are:

6.6.1. Union of Sets

For two given sets A and B, A∪B (read as A union B) is the set of distinct elements that belong to set A and set B or both. The number of elements in A ∪ B is given by n(A∪B) = n(A) + n(B) − n(A∩B), where n(X) is the number of elements in set X. To understand this set operation of the union of sets better, let us consider an example: If A = {1, 2, 3, 4} and B = {4, 5, 6, 7}, then the union of A and B is given by A ∪ B = {1, 2, 3, 4, 5, 6, 7}.

6.6.2. Intersection of Sets

For two given sets A and B, A∩B (read as A intersection B) is the set of common elements that belong to set A and B. The number of elements in A∩B is given by

n(A∩B) = n(A) + n(B) − n(A∪B), where n(X) is the number of elements in set X. To understand this set operation of the intersection of sets better, let us consider an example:

If A = {1, 2, 3, 4} and B = {3, 4, 5, 7}, then the intersection of A and B is given by

A ∩ B = {3, 4}.

6.6.3 Difference of two set

The set operation difference between sets implies subtracting the elements from a set which is similar to the concept of the difference between numbers. The difference between sets A and set B denoted as A − B lists all the elements that are in set A but not in set B. To understand this set operation of set difference better, let us consider an example: If A = {1, 2, 3, 4} and B = {3, 4, 5, 7}, then the difference between sets A and B is given by A - B = {1, 2}.

6.6.4. Complement of Sets

The complement of a set A denoted as A′ or A^c (read as A complement) is defined as the set of all the elements in the given universal set(U) that are not present in set A.

To understand this set operation of complement of sets better, let us consider an example: If U = {1, 2, 3, 4, 5, 6, 7, 8, 9} and A = {1, 2, 3, 4}, then the complement of set A is given by A' = {5, 6, 7, 8, 9}.

6.7. Properties of Set Operations

The properties of set operations are similar to the properties of fundamental operations on numbers. The important properties on set operations are stated below:

- Commutative Law - For any two given sets A and B, the commutative property is defined as,

  A ∪ B = B ∪ A

  This means that the set operation of union of two sets is commutative.

  A ∩ B = B ∩ A

  This means that the set operation of intersection of two sets is commutative.
- Associative Law - For any three given sets A, B and C the associative property is defined as,

  (A ∪ B) ∪ C = A ∪ (B ∪ C)

  This means the set operation of the union of sets is associative.

  (A ∩ B) ∩ C = A ∩ (B ∩ C)

  This means the set operation of intersection of sets is associative.
- De-Morgan's Law - The De Morgan's law states that for any two sets A and B, we have (A ∪ B)' = A' ∩ B' and (A ∩ B)' = A' ∪ B'
- A ∪ A = A
- A ∩ A = A
- A ∩ ∅ = ∅
- A ∪∅ = A

- $A \cap B \subseteq A$
- $A \subseteq A \cup B$

6.8. Laws of Set Operations

Here we will learn about some of the laws of algebra of sets.

1. Commutative Laws:

For any two finite sets A and B;

(i) A U B = B U A

(ii) A ∩ B = B ∩ A

2. Associative Laws:

For any three finite sets A, B and C;

(i) (A U B) U C = A U (B U C)

(ii) (A ∩ B) ∩ C = A ∩ (B ∩ C)

Thus, union and intersection are associative.

3. Idempotent Laws:

For any finite set A;

(i) A U A = A

(ii) A ∩ A = A

4. Distributive Laws:

For any three finite sets A, B and C;

(i) A U (B ∩ C) = (A U B) ∩ (A U C)

(ii) A ∩ (B U C) = (A ∩ B) U (A ∩ C)

Thus, union and intersection are distributive over intersection and union respectively.

5. De Morgan's Laws:

For any two finite sets A and B;

(i) A – (B U C) = (A – B) ∩ (A – C)

(ii) A - (B ∩ C) = (A – B) U (A – C)

De Morgan's Laws can also be written as:

(i) (A U B)' = A' ∩ B'

(ii) (A ∩ B)' = A' U B'

- More laws of algebra of sets:

6. For any two finite sets A and B;

(i) A – B = A ∩ B'

(ii) B – A = B ∩ A'

(iii) A – B = A ⇔ A ∩ B = ∅

(iv) (A – B) U B = A U B

(v) (A – B) ∩ B = ∅

(vi) A⊆ B ⇔ B' ⊆ A'

(vii) (A – B) U (B – A) = (A U B) – (A ∩ B)

7. For any three finite sets A, B and C;

(i) A – (B ∩ C) = (A – B) U (A – C)

(ii) A – (B U C) = (A – B) ∩ (A – C)

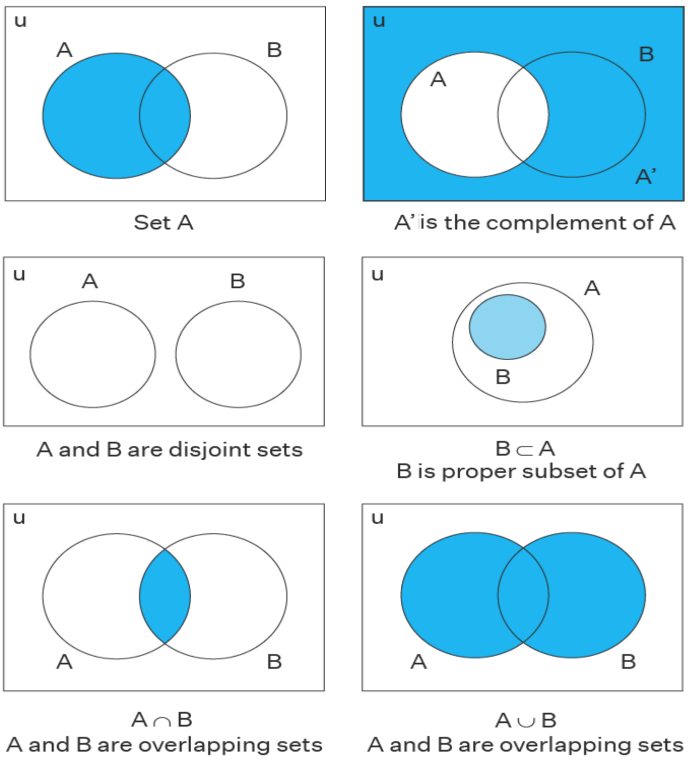
(iii) A ∩ (B - C) = (A ∩ B) - (A ∩ C)

(iv) A ∩ (B △ C) = (A ∩ B) △ (A ∩ C)

6.9. Venn Diagram

The graphic representation of the distinctions and affinities between two concepts is a Venn diagram. Venn diagrams are frequently used in set theory, logic, mathematics, companies, education, computer science, and statistics. They are sometimes known as logic or set diagrams.

Some Venn Diagram



Application of Venn diagram

- The use of Venn diagrams has a number of benefits. In various disciplines, including statistics, linguistics, logic, education, computer science, and business, venn diagrams are used to represent concepts and groupings.
- Information can be arranged visually to show the connections between groups of elements, such as similarities and differences, and to represent the connections for visual communication.
- Comparing two or more subjects allows us to clearly distinguish between their similarities and differences. This could be done while choosing a crucial good or service to purchase.
- Venn diagrams are another tool that mathematicians employ to solve challenging equations.

6.10. Summary

- ✔ Set symbol { }
- ✔ Empty set- ∅
- ✔ Universal set U
- ✔ Subset ⊆
- ✔ Union of two set A U B (A or B)
- ✔ Intersection of two set A ∩ B (A and B)
- ✔ Complement of set A'
- ✔ Demorgan's law i) (A U B)' = A' ∩ B' ii) (A ∩ B)' = A' U B'

6.11. Self-Assessment Questions

1. Write down any two empty sets.
2. Express BEAUTIFUL in set notations.
3. Let A = {1, 3, 5} and B = {1, 3, 5, 7}. Check whether A is subset of B or B is subset of A
4. Which of the following sets are null sets

   (a) {x: |x |< -4, x €N} (b) 2 and 3

   (c) Set of all prime numbers between 15 and 19 (d) {x: x < 5, x > 6}
5. Which of the following two sets are equal?

   (a) A = {1, 2} and B = {1} (b) A = {1, 2} and B = {1, 2, 3}

   (c) A = {1, 2, 3} and B = {2, 1, 3} (d) A = {1, 2, 4} and B = {1, 2, 3}

6. If A = {1, 3, 5, 7, 9} and B = {2, 3, 5, 7}, Then find $A \cup B$
7. Whether it is a True or False statement. Set of odd natural numbers divisible by 2 is a null set
8. Whether it is True or False statement .Set of even prime numbers is not an null set
9. If A = {1,2,3}, B= {3,4}, C= {4,5,6} then $A \cup (B \cap C)$ is
10. If A,B,C are the sets for the letters in the words college, marriage, luggage respectively. Then verify that [A – (B ∪ C) = [(A – B) ∩ (A – C)]
11. Write application of Venn Diagram.

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Unit 7

Determinants

Learning Objectives

After studying the unit, students will be able to:

- Solving determinant of order 2, 3 and 4
- Properties of determinant
- Operations on determinant

Structure

7.1. Introduction

7.2. Determinant of order 2,3,4

7.3. Multiplication of Determinant

7.4. Properties of Determinant

7.5. Summary

7.6. Self-assessment questions

7.7. Reference

7.1. Introduction

Determinants, in accordance with a predetermined rule, are the scalar numbers formed by adding the products of the square matrix's constituent elements and their cofactors. They aid in determining a matrix's adjoint, inverse. We must use this idea in order to solve linear equations using the matrix inversion approach. By computing determinants, it is simple to recall the cross-product of two vectors.

Learn more about how to identify determinants of various orders, their characteristics, and how to solve a few cases here.

Determinants are regarded as a matrix scaling factor. They can be viewed as results of the matrices' expansion and contraction. Determinants provide a single integer as their output after receiving a square matrix as their input.

For every square matrix, C = [cij] of order n×n, a determinant can be defined as a scalar value that is real or a complex number, where cij is the $(i, j)^{th}$ element of matrix C. The determinant can be denoted as det(c) or |C|, here the determinant is written by taking the grid of numbers and arranging them inside the absolute-value bars instead of using square brackets.

Consider a matrix

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Then, its determinant can be shown as:

$$|C| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

7.2. Determinant of order 2 , 3 and 4

7.2.1 Determinant of order 2

For any 2x2 square matrix or a square matrix of order 2×2, we can use the determinant formula to calculate its determinant:

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Its 2x2 determinant can be calculated as:

$$|C| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (a\times d) - (b\times c)$$

$$\text{For example: } C = \begin{bmatrix} 8 & 6 \\ 3 & 4 \end{bmatrix}$$

Its determinant can be calculated as:

$$|C| = \begin{vmatrix} 8 & 6 \\ 3 & 4 \end{vmatrix}$$

|C| = (8×4) - (6×3) = 32 - 18 = 14

7.2.2. Determinant of order 3

For any 3x3 square matrix or a square matrix of order 3×3,

$$C = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},$$ the determinant is represented as:

$$|C| \text{ (or) } \det C = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Here are the steps in calculating the determinant of a 3x3 matrix.

- a_1 is fixed as the anchor number and the 2x2 determinant of its sub-matrix (minor of a_1).
- Similarly, calculate the minors of b_1 and c_1.
- Keep multiplying the small determinant by the anchor number and by its sign

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Finally sum them up.

$$|C| = a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \cdot \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \cdot \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

|C|= a1(b2c3−b3c2)−b1(a2c3−a3c2)+c1(a2b3−a3b2)

Let's consider this example:

$$B = \begin{bmatrix} 3 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

Its determinant is calculated as:

$$|B| = \begin{vmatrix} 3 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{vmatrix}$$

$$= 3 \cdot \begin{vmatrix} -2 & 5 \\ 8 & 7 \end{vmatrix} - 1 \cdot \begin{vmatrix} 4 & 5 \\ 2 & 7 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & -2 \\ 2 & 8 \end{vmatrix}$$

= 3 × ((-2)(7) - (5)(8)) -1 × ((4)(7) - (5)(2)) + 1 × ((4)(8) - (-2)(2))

= 3 × ((-14) - (40)) -1 × ((28) - (10)) + 1 × ((32) - (-4))

= 3 × (-54) -1 × (18) + 1 × (36)

= - 162 - 18 + 36

= -144

Take note that we have used the first row here to determine the determinant of a 3x3 matrix. But the determinants may be calculated using any row/column.

7.2.3 Determinant of a 4×4 Matrix

Consider the 4x4 square matrix shown below or a square matrix of order 4×4. The following modifications should be taken into account when determining the determinant of a 4×4 matrix:

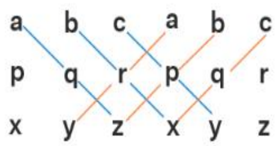
$$B = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}$$

- plus a_1 times the determinant of the 3x3 matrix obtained by deleting the row and the column containing a_1
- minus b_1 times the determinant of the 3x3 matrix obtained by deleting the row and the column containing b_1
- plus c_1 times the determinant of the 3x3 matrix obtained by deleting the row and the column containing c_1
- minus d_1 times the determinant of the 3x3 matrix obtained by deleting the row and the column containing d_1

$$|B| = a_1 \cdot \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1 \cdot \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix}$$

$$+ c_1 \cdot \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1 \cdot \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}$$

We can use the method mentioned in the previous section to find the determinant of the 3×3 matrices. Here is an easy way of finding it.



$$|A| \text{ (or) } \det A = aqz + brx + cpy - ary - bpz - cqx$$

7.3. Multiplication of 3×3 Determinants

Consider two matrices C and D of order 3×3, we first denote their respective determinants as |C| and |D| as shown below:

$$|C| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$|D| = \begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}$$

$$|C| \times |D| =$$

$$\begin{vmatrix} a_1p_1 + b_1p_2 + c_1p_3 & a_1q_1 + b_1q_2 + c_1q_3 & a_1r_1 + b_1r_2 + c_1r_3 \\ a_2p_1 + b_2p_2 + c_2p_3 & a_2q_1 + b_2q_2 + c_2q_3 & a_2r_1 + b_2r_2 + c_2r_3 \\ a_3p_1 + b_3p_2 + c_3p_3 & a_3q_1 + b_3q_2 + c_3q_3 & a_3r_1 + b_3r_2 + c_3r_3 \end{vmatrix}$$

These are some of the points to be remembered while multiplying two determinants:

- In order to multiply two determinants, we need to make sure that both are of the same order
- The value of the determinant does not change when rows and columns are interchanged, so we can also follow column by row, row by row, or column by column multiplication rules to multiply two determinants.

7.4. Properties of Determinants

For square matrices of different types, when its determinant is calculated, they are calculated based on certain important properties of the determinants. Here is the list of some of the important properties of the determinants:

Property1: "The determinant of an identity matrix is always 1.

Property 2: "If any square matrix B with order n×n has a zero row or a zero column, then det(B) = 0"

Property 3: "If C is upper or a lower-triangular matrix, then det(C) is the product of all its diagonal entries".

Property 4: "If D is a square matrix, then if its row is multiplied by a constant k, then the constant can be taken out of the determinant"

Other important properties of determinants are:

- A square matrix C is considered to be invertible if and only if det(C) ≠ 0.
- If B and C are two square matrices with order n × n, then det(BC) = det(B) × det(C) = det(C) × det(B)
- The relationship between a determinant of a matrix D and its adjoint adj(D) can be shown as D × adj(D) = adj(D) × D = |D| × I. Here, D is a square matrix and I is an identity matrix.

7.4.1. Rules For Operations on Determinant

The following rules are helpful to perform the row and column operations on determinants.

- If the rows and columns are switched, the determinant's value is unaffected.
- If any two rows or (two columns) are switched, the determinant's sign changes.
- Any two equal rows or columns in a matrix result in the determinant's value being zero.
- The value of the determinant is multiplied by the constant if each element in a given row or column is multiplied by the constant.
- The determinant can be stated as a sum of determinants if the elements of a row or column are expressed as a sum of elements.
- The value of the determinant is unaffected if the elements of one row or column are added to or subtracted from the equivalent multiples of elements in another row or column.

7.4.2. Other Properties of Determinants

To get the value of the determinant with the fewest calculations, properties of determinants are required. Determinants' characteristics are based on element, row, and column operations, which makes it simple to get the determinant's value.

Here, we will study more about determinant qualities and work through various cases that have been solved to help us better comprehend the idea.

The characteristics of determinants are useful for quickly and with the fewest computations computing the value of the determinant. The following are the seven crucial determining qualities.

1. Interchange Property

If the rows or the columns of a determinant are switched, the value of the determinant is unaffected.

$$A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, A' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Det(A) = Det(A')

The determinant value and the determinant of the transposition are equivalent because of this characteristic, which means that if the rows and columns of the matrix are switched, the matrix is transposed.

2. Sign Property

The sign of the value of the determinant changes if any two rows or any two columns are interchanged.

$$A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, B = \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Det(A) = -Det(B)

Only when the row or column is switched once does the value of the determinant change the sign. Det(A) = -Det(B) is the result of swapping the second row with the third row in the above matrix A to create matrix B. If the value of the determinant is D, and the rows or columns are swapped n times, then the new value of the determinant is $(-1)^n D$.

3. Zero Property

The value of a determinant is equal to zero if any two rows or any two columns have the same elements.

$$A = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here the elements of the first row and the second row are identical. Hence the value of the determinant is equal to zero.

Det(A) = 0

4. Multiplication Property

The value of the determinant becomes k times the earlier value of the determinant if each of the elements of a particular row or column is multiplied with a constant k.

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, B = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Det(B) = k× Det(B)

The determinant value is multiplied by a constant k, and the elements in the first row are multiplied by the same constant k. With the use of this characteristic, a common factor may be extracted from each row or column of the determinant. Additionally, the value of the determinant is equal to zero if any two rows or columns have matching components.

5. Sum Property

The determinant can be stated as the sum of two or more determinants if a few items of a row or column are expressed as a sum of terms.

$$\begin{vmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

Term sums that may be divided into two distinct determinants are represented by the items in the first row. Additionally, the new determinants share the same second and third rows as the prior determinant.

6. Property Of Invariance

The value of the determinant is unaffected if each element of a row and column of a determinant is added with the equimultiples of the elements of another row or column of a determinant.

This can be expressed in the form of a formula as $R_i \rightarrow R_i + kR_j$, or $C_i \rightarrow C_i + kC_j$.

$$A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, B = \begin{vmatrix} a_1 + kc_1 & a_2 + kc_2 & a_3 + kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Det(A) = Det(B)

The first row of matrix A's elements were swapped out for the total of those elements, and the third row was multiplied by a constant to create the new matrix B. The determinant A in this case is equivalent to the determinant B following this procedure.

7. Triangular Property

If the elements above or below the main diagonal are equal to zero, then the value of the determinant is equal to the product of the elements of the diagonal matrix.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

7.5. Summary

- ✔ Determinant of order 2 has 2 rows and 2 columns
- ✔ Determinant of order 3 has 3 rows and 3 columns
- ✔ Determinant of order 4 has 4 rows and 4 columns
- ✔ Property 1- The value of the determinant remains unchanged if the rows and columns of a determinant are interchanged.
- ✔ Property 2- If any two rows (or columns) of determinants are interchanged, then sign of determinants changes.
- ✔ Property 3- If any two rows or columns of a determinant are equal or identical, then the value of the determinant is 0.
- ✔ Property 4- If each element of a row or a column is multiplied by a constant value k, then the value of the determinant originally obtained is multiplied with k.

- The value of the determinant is unaffected if each element of a row and column of a determinant is added with the equimultiples of the elements of another row or column of a determinant
- A group of points is said to be convex if and only if the line AB connecting any two points A, B in the set completely encircles the set.
- Quasiconvexity is an extension of convexity since all convex functions are also quasiconvex, but not all quasiconvex functions are convex

7.6. Self-assessment questions

1. If $|2x - 142| = |3021|$ then find x

   a) 3 b)2/3 c)3/2 d) -1/4
2. Determinant is a number associated with a matrix. Choose correct option

   a) True b) False c) May be true or may be false d) none
3. Evaluate $|sin30cos30 - sin60co60|$
4. Show that points A(a, b + c), B(b, c + a) and C(c, a + b) are collinear
5. The value of the determinant$|2345686x9x12x|$
6. By using properties of determinants,

   prove that $|0b - cc - ac - b0b - aa - ca - b0|$= 0
7. Prove that $|1aa^2 1bb^2 1cc^2|$= $|1abc1bca1cab|$
8. Write properties of determinant
9. Explain 2*2 determinant with example
10. Explain 3*3 determinant with example

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- https://www.chilimath.com/lessons/advanced-algebra/determinant-3x3-matrix/

Unit 8

Matrices

Learning Objectives

After studying the unit, students will be able to:

- Introduction and types of matrices
- Operations on matrices
- Various properties of matrices
- Find inverse of matrix
- Understand the Concept of rank of matrix

Structure

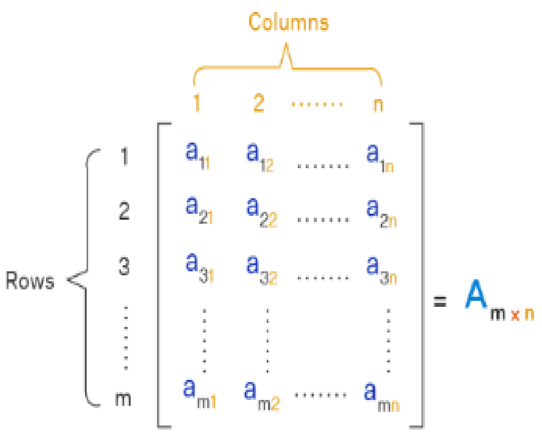
8.1. Introduction

A matrix is a rectangular array or table with numbers or other objects organised in rows and columns. Matrices is the plural version of matrix. The number of columns and rows is unlimited. Matrix operations include addition, scalar multiplication, multiplication, transposition, and many others.

There are some guidelines to follow while conducting these matrix operations, such as the fact that they can only be multiplied if the first and second sets of columns are precisely the same, while they can only be added or subtracted if they have the same number of rows and columns. Let's take a closer look at these laws and the many kinds of matrices. The arrangements of numbers, variables, symbols, or phrases in a rectangular table with varying numbers of rows and columns are called matrices, the plural version of the word matrix. They are arrays that are rectangular in shape, and addition, multiplication, and transposition are just a few of the defined operations. The constituents of the matrix are its digits or entries. Vertical elements in matrices are known as columns, whereas horizontal entries are referred to as rows.

A matrix is a rectangular array of values that are specified for mathematical operations including addition, subtraction, and multiplication. The quantity of rows and columns in a matrix—also referred to as the matrix's order—determines the matrix's size. When written as a 6×× 4 and read as 6 by 4, the order of a matrix with 6 rows and 4 columns is displayed. For instance, the given matrix B is a 3 ×4 matrix and is represented by the notation [B]3×4:

$$B = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 5 & 2 & 7 \\ 1 & -1 & -2 & 9 \end{bmatrix}$$



- **Notation of Matrices**

A matrix will contain m n items if it has m rows and n columns. A matrix is denoted by an uppercase letter, in this example the letter "A," and its elements are denoted by a lowercase letter and two subscripts, in this case the letters "aij" which indicate the element's location in the matrix's rows and columns in the same order. For instance, the element in the third row and second column of the provided matrix A would be a32 as seen in the matrix shown below:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33}\ldots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3}\ldots & a_{mn} \end{bmatrix}$$

8.2 Types of Matrices:

1) Row matrix: Matrix having only one row eg. $[\ 2\ ]_{1\times 1}$ $[2\ \ 3\ \ 5\ ]_{1\times 3}$

   Denoted by - $1\times n$

2) Column matrix: A matrix having only one column.

   Eg. $[\ 3\ ]_{1\times 1}$ $\begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} 3\times 1$

   Denoted by – $m\times 1$

3) Square matrix: A matrix in which no. of rows is equal to no. of column.

$$\begin{bmatrix} 2 & 3 \\ 0 & 7 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 7 \\ 1 & 8 & -2 \end{bmatrix} 3\times 3$$

4) Diagonal matrix: The square matrix in which all non – diagonal elements are zero

E.g. -: $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

5) Scalar matrix: The Diagonal matrix in which all diagonal elements are equal

E.g. -: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

6) Identity matrix: (Unit matrix)

The scalar matrix in which all diagonal elements are unity. Denoted by 'I'

Eg -: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7) Null matrix : (Zero matrix)

A matrix having all elements is zero.

Eg -: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

8) Upper Triangular matrix:

A square matrix in which all the elements below diagonal are zero.

E.g. -: $\begin{bmatrix} 2 & 3 \\ 0 & 7 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 7 \\ 0 & 5 & 8 \\ 0 & 0 & -1 \end{bmatrix}$

9) Lower Triangular matrix

A square matrix in which all the elements above diagonal are zero.

$\begin{bmatrix} 2 & 0 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 5 & 0 \\ 2 & 7 & -2 \end{bmatrix}$

10) Transpose of matrix:

Let A be a square matrix, then by interchanging rows & columns of matrix A.

Denoted by Transpose of A = A^1

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 7 \\ -2 & 5 \end{bmatrix} \quad A^1 = \begin{bmatrix} 3 & 1 & -2 \\ 0 & 7 & 5 \end{bmatrix}$$

11) Symmetric matrix:

The square matrix in which

a) $A = A^1$

b) $a_{ij} = a_{ji}$

$$\begin{bmatrix} 1 & 3 & -7 \\ 3 & 2 & 8 \\ -7 & 8 & 5 \end{bmatrix}$$

12) Skew – symmetric matrix:

The square matrix in which

a) $A = -A^1$

b) $a_{ij} = - a_{ji}$

c) All diagonal elements are zero.

$$\begin{bmatrix} 0 & 7 & -2 \\ -7 & 0 & 5 \\ 2 & -5 & 0 \end{bmatrix}$$

13) Hermitian Matrix: The square matrix in which A = (a_{ij}) is called

Determinant of matrix:

A square matrix has determinant

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad 1A1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - 6 = -2$$

14) Singular matrix:

A square matrix in which 1A1 = 0

$$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \quad 1A1 = 0$$

15) Non – singular matrix:

A square matrix in which 1A1 $\neq$ 0

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \quad 1A1 = 18 - 20 = -2 \neq 0$$

16) Orthogonal Matrix: Any square matrix A is orthogonal if its transpose is equal to its inverse. i.e., $A^T = A^{-1}$

8.3. Algebra of Matrices

By applying operations on matrices, like addition, subtraction, multiplication, and so forth, we may solve them. The quantity of rows and columns affects how matrices are calculated. The number of rows and columns must match for addition and subtraction, but for multiplication, the number of columns in the first matrix and the number of rows in the second matrix must match. The fundamental operations that may be carried out on matrices include:

- Addition of Matrices
- Subtraction of Matrices
- Scalar Multiplication
- Multiplication of Matrices
- Transpose of Matrices

8.3.1. Addition of Matrices

Only when two matrices have the same number of rows and columns can they be added together.

While adding 2 matrices, we add the corresponding elements. i.e., (A + B) = [aij] + [bij] where i and j are the number of rows and columns respectively. For example:

$$\begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & -1+2 \\ 0+1 & 5+(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

8.3.2. Subtraction of Matrices

Matrices can be subtracted, however it's only feasible if both matrices have the same amount of rows and columns. While subtracting 2 matrices, we subtract the corresponding elements. i.e., (A - B) = [aij] - [bij] where i and j are the row number and column number respectively. For example:

$$\begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0 & -1-2 \\ 0-1 & 5-(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 \\ -1 & 7 \end{bmatrix}$$

8.3.3. Scalar Multiplication

The product of a matrix A with any number 'c' is obtained by multiplying every entry of the matrix A by c, is called scalar multiplication. i.e., (cA)ij = c(Aij)

- **Properties of scalar multiplication in matrices**

The different properties of matrices for scalar multiplication of any scalars K and l, with matrices A and B are given as,

- K(A + B) = KA + KB
- (K + l)A = KA + lA
- (Kl)A = K(lA) = l(KA)
- (-K)A = -(KA) = K(-A)
- 1·A = A
- (-1)A = -A

8.3.4. Multiplication of Matrices

Only when two matrices have the same number of rows and columns can they be added together. To understand how matrices are multiplied, let us first consider a row vector R=[r1 r2...rn] and a column vector

C =

$$: \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Then the product of *R* and *C* can be

defined as RC = $[r1 \ \ r2 \ \ \dots \ rn] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

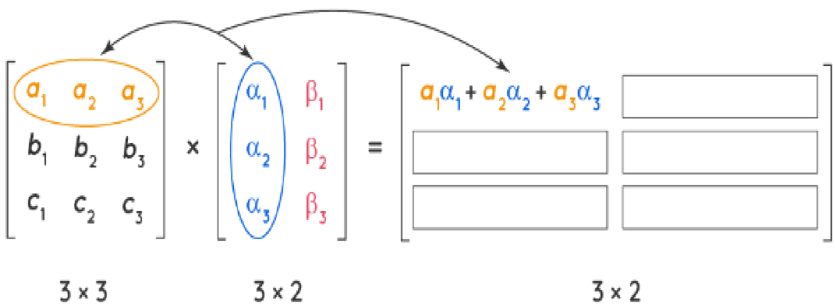
=[r1c1 + r2c2 +...+ rncn]

We'll talk about matrix multiplication now. It will quickly become clear that the number of columns in A must match the number of rows in B in order to multiply two matrices A and B and obtain AB.

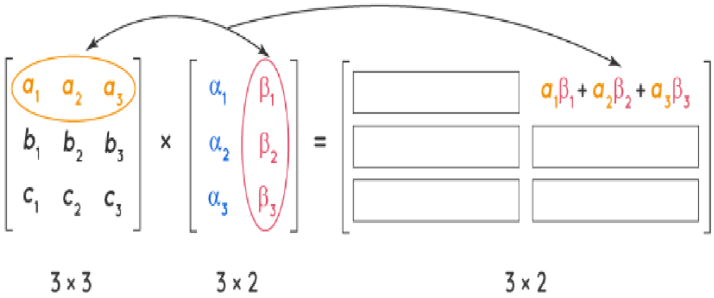
Let *A* be of order $m \times n$ and *B* be of order $n \times p$. The matrix *AB* will be of order $m \times p$ and will be obtained by multiplying each row vector of *A* successively with column vectors in *B*. Let us understand this using a concrete example:

$$:A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} B = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_1 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$$

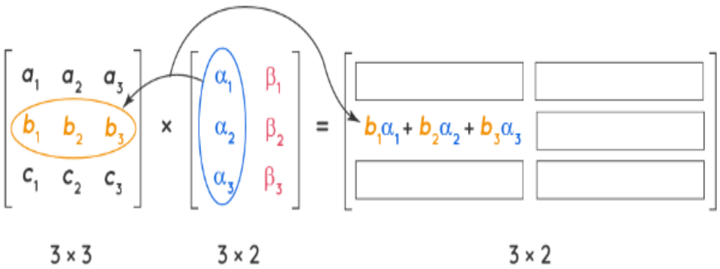
To obtain the element a11 of AB, we multiply R1 of A with C1 of B :



To obtain the element a12 of AB, we multiply R1 of A with C2 of B:

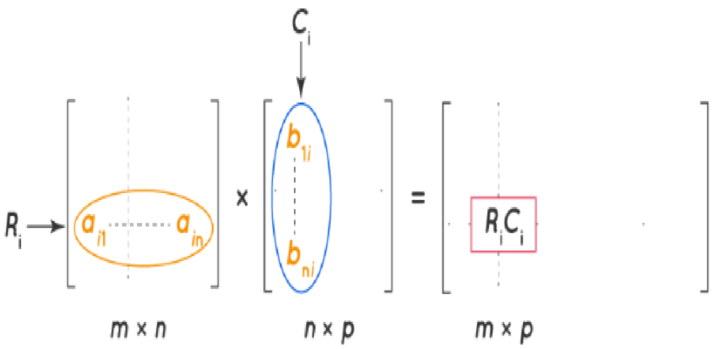


To obtain the element a21 of AB, we multiply R2 of A with C1 of B:



Proceeding this way, we obtain all the elements of AB.

Let us generalize this: if A is or order m × n, and B of order n × p, then to obtain the element aijaij in AB, we multiply RiRi in A with CjCj in B:

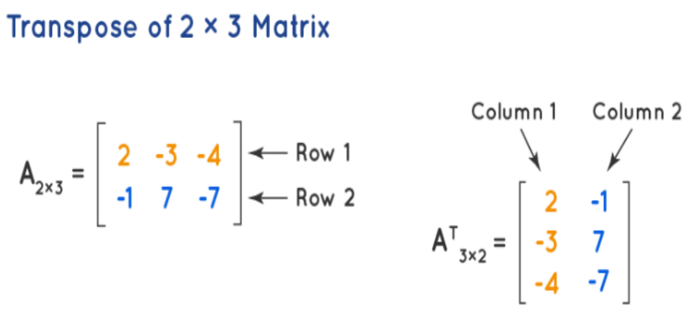


The multiplication of matrices is accompanied by a variety of features. Regardless of the three matrices A, B, or C:

- AB ≠ BA
- A(BC) = (AB)C
- A(B + C) = AB + AC
- (A + B)C = AC + BC
- AIm = A = AI_n, for identity matrices Im and I_n.
- Am×nOn×p = Om×p, where O is a null matrix.

8.3.5. Transpose of Matrix

When we swap out a matrix's rows for its columns and vice versa, we are said to have transposed the matrix. The transpose of matrices is the reversal of rows and columns. Row elements in the matrix below are row-1: 2, -3, -4, and row-2: -1, -7. In the graphic below, we can see that after transposing, we will receive the components in column 1: 2, -3, -4, and column 2: -1, 7, -7.



- Properties of transposition in matrices

There are various properties associated with transposition. For matrices A and B, given as,

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$, A and B being of the same order.
- $(KA)^T= KA^T$, K is any scalar(real or complex).
- $(AB)^T= B^TA^T$, A and B being conformable for the product AB. (This is also called reversal law.)

8.3.6. Trace of a Matrix

The trace of any matrix A, Tr(A) is defined as the sum of its diagonal elements. Some properties of trace of matrices are,

- tr(AB) = tr(BA)
- $tr(A) = tr(A^T)$
- tr(cA) = c tr(A), for a scalar 'c'
- tr(A + B) = tr(A) + tr(B)

8.3.7. Minor of Matrix

Minor for a particular element in the matrices is defined as the determinant of the matrix that is obtained when the row and column of the matrix in which that particular element lies are deleted, and the minor of the element aijaij is denoted as Mij. For example, for the given matrix, minor of a12 of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is:}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Similarly, we can find all the minors of the matrix and will get a minor matrix M of the given matrix A as:

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

8.3.8. Cofactor of Matrix

Cofactor of an element in the matrix A is obtained when the minor Mij of the matrix is multiplied with $(-1)^{i+j}$. The cofactor of a matrix is denoted as Cij. If the minor of a matrix is Mij then the cofactor of the matrix would be:

$Cij = (-1)^{i+j}Mij$

On finding all the cofactors of the matrix, we will get a cofactor matrix C of the given matrix A:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

8.3.9. Adjoint of Matrices

By determining the transpose of the cofactors of the elements in the provided matrices, the adjoint of matrices is calculated. We must first determine the cofactors of each matrix element before transposing the cofactor matrix to obtain the adjoint of the supplied matrix.

The adjoint of matrix A is denoted by adj(A). Let us understand this with an example: We have a matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 2 \\ 1 & -1 & -2 \end{bmatrix}$$

Then the minor matrix M of the given matrix would be:

$$M = \begin{bmatrix} -8 & -2 & -5 \\ 5 & -7 & -1 \\ -17 & 4 & 10 \end{bmatrix}$$

We will get the cofactor matrix C of the given matrix A as:

$$C = \begin{bmatrix} -8 & 2 & -5 \\ -5 & -7 & 1 \\ -17 & -4 & 10 \end{bmatrix}$$

Then the transpose of the cofactor matrix will give the adjoint of the given matrix

$$adj(A) = C^T = \begin{bmatrix} -8 & -5 & -17 \\ 2 & -7 & -4 \\ -5 & 1 & 10 \end{bmatrix}$$

8.4. Inverse of Matrices

Any matrix's inverse is represented by the matrix raised to the power (-1), hence the inverse matrix for any matrix "A" is represented as A^{-1}. The inverse of a square matrix, A is A^{-1} only when: $A \times A^{-1} = A^{-1} \times A = I$. There is a possibility that sometimes the inverse of a matrix does not exist if the determinant of the matrix is equal to zero ($|A| = 0$). The inverse of a matrix is shown by A^{-1}. Matrices inverse is calculated by using the following formula:

$$A^{-1} = \frac{1}{1A1} \text{ (adj of A)}$$

where

- $|A|$ is the determinant of the matrix A and $|A| \neq 0$.
- Adj A is the adjoint of the given matrix A.

The inverse of a 2 × 2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is calculated by $A^{-1} = \dfrac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

Let us find the inverse of the 3 × 3 matrix we have used in the previous section

$$: A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 2 \\ 1 & -1 & -2 \end{bmatrix}$$

$$\text{Since adj}(A) = \begin{bmatrix} -8 & -5 & -17 \\ 2 & -7 & -4 \\ -5 & 1 & 10 \end{bmatrix}$$

And on calculating the determinant, we will get |A| = -33

$$\text{Therefore, } A^{-1} = (1/-33) \times \begin{bmatrix} -8 & -5 & -17 \\ 2 & -7 & -4 \\ -5 & 1 & 10 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 0.24 & 0.15 & 0.51 \\ -0.06 & 0.21 & 0.12 \\ 0.15 & -0.03 & -0.39 \end{bmatrix}$$

- **Elementary Transformations-:**

There are two types of elementary transformations.

8.4.1. Elementary Row transformation-:

1) Interchange of any two rows. -: In general interchange of i^{th}&j^{th} row is denoted by

Ri ↔ Rj or (Rij)

2) Multiplying to each element of any one row by a non- zero scalar.

Multiplying to each element of i^{th} row by a non-zero scalar is denoted by K.Ri.

3) Adding to each element of any one row the non- zero multiples of the corresponding elements of any other row.

$Ri \rightarrow Ri + KRj$ means adding to each element of i^{th} row by k multiples of the corresponding element of j^{th} row.

8.4.2. Elementary column transformation -:

1) Interchange of any two columns -:

Example-: $Ci \leftrightarrow Cj$ or (Cij)

2) Multiplying to each element of any one column by a non- zero scalar.

Ex-: K Ci

3) Adding to each element of any one column the non – zero multiples of the corresponding elements of any other column. Ex-: Ci□Ci + K Cj

8.4.3. Inverse of square matrix by Elementary Transformations

Definition-:

Let A is a non-singular matrix $(|A|) \neq 0$ If there exist a matrix B such that AB =BA=I where I is unit matrix of the same order then matrix B is called an inverse of A. which is denoted by $B = A^{-1}$

$$AA^{-1} = A^{-1} A = I$$

Remark -:

1) Inverse of the matrix A exists if & only if A is non – singular matrix (1A1$\neq$ 0)

2) Inverse of matrix (non- singular) can be obtained by using either row or column transformations or by using Adjoint of the matrix A.

3) It should be clearly remembered that once we decide on using either row or column transformations, we must continue to use only rows transformation or only column transformations till we get the inverse.

4) For row transformation use $AA^{-1} = I$.

5) For column transformation use $A^{-1} A = I$

8.5. Rank of a Matrix

The greatest number of linearly independent row(or column) vectors in a matrix A is referred to as the matrix's rank. In other words, a matrix's rank will never be more than or equal to the sum of its columns or rows. Since a null matrix lacks any independent row or column vectors, its rank is 0.

- How to determine matrix rank using echelon formalism

(i) The starting element of each non-zero row must be 1.

(ii) Rows with zero elements must be placed below rows with non-zero elements.

(iii) subsequent non-zero rows must have more zeros in total than preceding non-zero rows; A simple technique can be used to quick

8.6. Summary

- MATRIX is a rectangular array or table with numbers or other objects organised in rows and columns. Matrices is the plural version of matrix. The number of columns and rows is unlimited. Matrix operations include addition, scalar multiplication, multiplication, transposition, and many others.
- TRANSPOSE OF A MATRIX is an operator which flips a matrix over its diagonal; that is, it switches the row and column indices of the matrix A by producing another matrix, often denoted by A^T.
- DETERMINANT OF A MATRIX determined by adding the products of each element in a given row or column and their corresponding cofactors. The symbol for the determinant of a matrix A is lAl.
- VECTORS is a quantity or phenomenon that has two independent properties: magnitude and direction. The term also denotes the mathematical or geometrical representation of such a quantity.
- NEGATIVE VECTORS are two vectors that have the same magnitude but move in the opposite directions, they are said to be the negative of one another. Vector A is referred to as the negative of vector B or vice versa if vectors A and B have the same magnitude but
- move in the opposite directions.
- PARALLEL VECTORS are when two or more vectors have the same direction but not necessarily the same magnitude, they are said to be parallel. Parallel vectors'

direction angles differ by zero degrees. Antiparallel vectors are those whose angle of direction differs

- by 180 degrees; in other words, antiparallel vectors have the opposite directions.
- There are different formulas associated with matrix operations depending upon the type of matrix. Some of the matrices formulas are listed below:
- $A(\text{adj } A) = (\text{adj } A) A = | A | I_n$
- $| \text{adj } A | = | A |^{n-1}$
- $\text{adj } (\text{adj } A) = | A |^{n-2} A$
- $| \text{adj } (\text{adj } A) | = | A |^{(n-1)\text{^}2}$
- $\text{adj } (AB) = (\text{adj } B) (\text{adj } A)$
- $\text{adj } (A^m) = (\text{adj } A)^m$,
- $\text{adj } (kA) = k^{n-1} (\text{adj } A)$, $k \in R$
- $\text{adj}(I_n) = I_n$
- adj 0 = 0
- A is symmetric ⇒ (adj A) is also symmetric.
- A is diagonal ⇒ (adj A) is also diagonal.
- A is triangular ⇒adj A is also triangular.
- A is singular ⇒| adj A | = 0
- $A^{-1} = (1/|A|) \text{ adj } A$
- $(AB)^{-1} = B^{-1}A^{-1}$

8.7. Self-Assessment questions

1. $A= \begin{bmatrix} 2 & -1 & 3 \\ 2 & 4 & 6 \\ 3 & 1 & 4 \end{bmatrix}$ then find A) minor of $a_{11,}$ a_{23}, a_{33}

B) Cofactor of c_{12}, c_{23}, c_{32}, c_{31}

A) 10, 5, -18, 10

ANS:-

B) 10, -5, -6, -18

2. Find the adjoint of the following matrices.

1) $\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$ ANS-: $\begin{bmatrix} 1 & -4 \\ -1 & 2 \end{bmatrix}$

2) $\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$ ANS-: $\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$

3.Find the inverse of the following matrices by using their adjoint matrix

1) $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ ANS-: $\frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

2) $\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$ ANS-: $\frac{1}{5}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

3) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ ANS-: $\begin{bmatrix} 2 & 1 & -1 \\ 2 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

4) $\begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & -3 \\ 1 & 2 & 4 \end{bmatrix}$ ANS-: $\begin{bmatrix} -6 & 1\grave{}6 & 9 \\ 7 & -18 & -10 \\ -2 & 5 & 3 \end{bmatrix}$

5) $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 1 \end{bmatrix}$ ANS-: $\frac{1}{5}\begin{bmatrix} 4 & 1 & -4 \\ 3 & 2 & -8 \\ -1 & 1 & 1 \end{bmatrix}$

4.Find the inverse of the following matrices-:

1) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ Ans-: $\begin{bmatrix} 13 & 2 & -7 \\ -3 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

2) $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 1 \end{bmatrix}$ Ans-: $\frac{1}{5}\begin{bmatrix} 4 & 1 & -4 \\ 3 & 2 & -8 \\ -1 & 1 & 1 \end{bmatrix}$

5. If $A^T = [34 - 1201]$ and B=$[-121123]$ then find A^T - B^T

6. If the matrix A=$[0a - 320 - 1b10]$ is skew symmetric, find a and b.

7. If A=$[cos \propto sin \propto sin \propto cos \propto]$ and A+A' = I find $\propto$.

8.If A is a square matrix such that $A^2 = A$, then $(I + A)^2 - 3A$ is

A) I B) 2A C) 3I D) A

9. If A =$[5xy0]$ and A = A' then

A) x=0, y=5 B) x = y C) x + y = 5 D) x – y = 5

10.Find x, y, z if { $4[1203 - 12]$ - $[3214 - 24]$ }$[02]$ =$[xyz]$

11. What are Matrices? What is the transpose of a Matrix? Explain the trace of a matrix?

12 What is the minor and cofactor of a Matrix?

13. State the types of Matrices.

13 Define symmetric and skew symmetric Matrices.

14 What is an invertible Matrix?

8.8. Reference

- ✔ Strang, G. (2006): Linear Algebra and its Applications, Thomson Brooks/Cole
- ✔ Alpha C. Chiang, Kevin Wainwright (2005): Fundamental Methods of Mathematical Economics, 4th Edition, McGraw-Hill.http://www.khullakitab.com/matrices-and-determinants/notes/mathematics/grade-11/18/solutions

Unit 9

Relations And Functions

Learning Outcomes:

After studying the unit, students will be able to:

- Study types of relation
- Study types of function
- Understand ordered pairs
- Learn concept of two or more independent variable using partial derivative

Structure

9.1 Introduction

9.2. Laws of Relation Operations

9.3. Relation and Function

9.4. Types of Relations and Functions

9.5. Ordered Pairs

9.6. Summary

9.7. Self-assessment questions

9.8. Reference

9.1 Introduction

In real life, relations and functions provide the connection between any two things. We encounter various patterns and connections in daily life that define relationships, such as the relationship between a brother and sister or a father and son. We encounter several relationships between numbers in mathematics as well, such as a number x being less than y, a line l being parallel to a line m, etc. Elements of one set (domain) are mapped to elements of another set through relation and function (codomain).

Functions are only particular varieties of relations that specify the precise relationship between two quantities. In this post, we'll look at the many types of relations and functions, as well as how to connect pairs of items from two sets and construct a connection between them.

9.2. Laws of Relation Operations

We have already discussed what the relation exactly means. Relation is the term which shows the connection between different objects or elements or members, which further shows us some of the essentially implied linkages between different sets, and also the elements of the sets. So now it's the turn to know something more about the operations related to the relations.

The operations of relations can be studied in a better way, if we discuss the operations with reference to the types of the relations that we have, as different types of relations have different concepts and different workings, so as to make them uniquely different from each other. Some of them will be discussed below.

9.2.1. Empty Relation

As the name suggests itself, that is what the term may mean or define itself as. Empty Relation is nothing but just the relation where there are no such elements or objects or members are present. In simple words, it may be more clear, if one says an empty relation is defined when there are no elements present on the set that relate with each other in one way or the other. The other well-known name of the empty relation is widely known as the void relation, where void means empty.

This will become more clear, when we take the help of an example. So, let us consider, a set, names as set A, where,

Set A = { 3,4,5 }, where one can write the empty relation as, R, where we have,

R = { x,y },which gives us, |x-y| = 7, thus, if we relate the empty relation with the set by solving it, we will never get the condition that is set up for that particular set.

This gives us the rule of an empty relation as,
R = phi subset of Set A x Set A, where, R is the relation which is equivalent to the empty set, which is again is a subset of the products of the elements of the given set itself.

9.2.2. Universal Relation

An universal relation is that type of relation where every other element has a relation with all other elements. In simple words, one can define it as a relation where all the elements of the given sets are universally related, where universally means as a whole, all together. In an universal relation, what happens is that, if one set of elements are given, then if all the elements are related with each other in that set itself, then it will be considered as the universal. And, if there are two sets given, then if there is a possibility that all the elements of one set have relation with each and every element of the other set, then also it will be called a universal relation.
For any relation R and any givens et A, the rule of universal relation is given as,
R = A x A.

9.2.3. Reflexive Relation

As the name itself suggests, what the term may mean exactly. Reflexive means reflecting the same thing. In the relation also, it fits with the same thing. In reflexive function, what we have is that, the elements in any one of the given sets are those elements that are related with the elements of the original set itself. This rule is written as, for given elements as “a”, for the given set S,having the relation as R, then we get,
(a, a) belongs to R for all a belongs to S.

9.2.4. Symmetric Relation

Symmetric relation is the easiest type of relation to be understood, as this type of relation focuses on making us understand that, if there are any of the two sets given, then if the elements of the first are related to the elements of the second set, that is in the co-domain, then the reverse is also true, as the elements of the second set are also related to the elements of the first one. The rule can be written as, (a, b) belongs to R => (b, a) belongs to R.

9.2.5. Transitive Relation

In this type of relation, we get to see that, when the elements of one of the sets are being given and the relation is given as R, then the relation must …
X that contains all the elements, say c, that can be related to a.

- **What are Relations and Functions?**

In order to have ordered pairs of the kind, relations and functions provide a mapping between two sets (Inputs and Outputs) (Input, Output). In algebra, relationship and function are crucial ideas. Both in real life and in mathematics, they are often used. To further comprehend the significance of each of these relational and functional concepts, let's describe them.

9.3. Relation and Function

Relation and function individually are defined as:

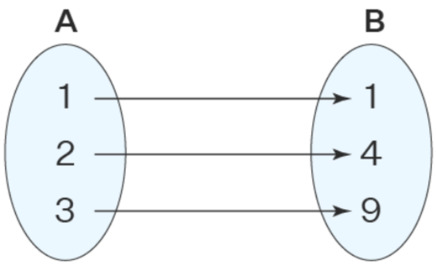
- Relations - A relation R from a non-empty set B is a subset of the cartesian product A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B.
- Functions - A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B. In other words, no two distinct elements of B have the same pre-image.

9.3.1. Representation of Relation and Function

Relations and functions can be represented in different forms such as arrow representation, algebraic form, set-builder form, graphically, roster form, and tabular form.

Define a function f: A = {1, 2, 3} → B = {1, 4, 9} such that f(1) = 1, f(2) = 4, f(3) = 9. Now, let us represent this function in different forms.

- Set-builder form - {(x, y): f(x) = y^2, x ∈ A, y ∈ B}
- Roster form - {(1, 1), (2, 4), (3, 9)}
- Arrow Representation –



- **Table Representation** -

x	**y**
1	1
2	4
3	9

9.3.2. Difference between Relation and Function

The fundamental distinction between a relation and a function is that a relation allows for the possibility of several outputs from a single input. In contrast, a function only has one output for each input. The distinctions between relations and functions are shown in the table that follows.

Relation	Function
A relation in math is a set of ordered pairs defining the relation between two sets.	A function is a relation in math such that each element of the domain is related to a single element in the codomain.
A relation may or may not be a function.	All functions are relations.
Example: {(1, x), (1, y), (4, z)}	Example: {(1, x), (6, y), (4, z)}

9.3.3. Terms Related to Relations and Functions

Now that we have understood the meaning of relation and function, let us understand the meanings of a few terms related to relations and functions that will help to understand the concept in a better way:

- **Cartesian Product** - Given two non-empty sets P and Q, the cartesian product $P \times Q$ is the set of all ordered pairs of elements from P and Q, that is, $P \times Q = \{(p, q) : p \in P, q \in Q\}$
- **Domain** - The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R. It is called the set of inputs or pre-images.
- **Range** - The set of all second elements of the ordered pairs in a relation R from a set A to a set B is called the range of the relation R. It is called the set of outputs or images.
- **Codomain** - The whole set B in a relation R from a set A to a set B is called the codomain of the relation R. Range $\subseteq$ Codomain.

9.4. Types of Relations and Functions

There are different types of relations and functions that have specific properties which make them different and unique. Let us go through the list of types of relations and functions given below:

9.4.1. Types of Relations

Given below is a list of different types of relations:

- Empty Relation - A relation is an empty relation if it has no elements, that is, no element of set A is mapped or linked to any element of A. It is denoted by R = ∅.
- Universal Relation - A relation R in a set A is a universal relation if each element of A is related to every element of A, i.e., R = A × A. It is called the full relation.
- Identity Relation - A relation R on A is said to be an identity relation if each element of A is related to itself, that is, R = {(a, a) : for all a ∈ A}
- Inverse Relation - Define R to be a relation from set P to set Q i.e., R ∈ P × Q. The relation R^{-1} is said to be an Inverse relation if R^{-1} from set Q to P is denoted by R^{-1} = {(q, p): (p, q) ∈ R}.
- Reflexive Relation - A binary relation R defined on a set A is said to be reflexive if, for every element a ∈ A, we have aRa, that is, (a, a) ∈ R.
- Symmetric Relation - A binary relation R defined on a set A is said to be symmetric if and only if, for elements a, b ∈ A, we have aRb, that is, (a, b) ∈ R, then we must have bRa, that is, (b, a) ∈ R.
- Transitive Relation - A relation R is transitive if and only if (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R for a, b, c ∈ A
- Equivalence Relation - A relation R defined on a set A is said to be an equivalence relation if and only if it is reflexive, symmetric and transitive.

9.4.2. Types of Functions

Given below is a list of different types of functions:

- One-to-One Function - A function f: A → B is said to be one-to-one if each element of A is mapped to a distinct element of B. It is also known as Injective Function.
- Onto Function - A function f: A → B is said to be onto, if every element of B is the image of some element of A under f, i.e, for every b ∈ B, there exists an element a in A such that f(a) = b. A function is onto if and only if the range of the function = B.
- Many to One Function - A many to one function is defined by the function f: A → B, such that more than one element of the set A are connected to the same element in the set B.
- Bijective Function - A function that is both one-to-one and onto function is called a bijective function.

- Constant Function - The constant function is of the form f(x) = K, where K is a real number. For the different values of the domain(x value), the same range value of K is obtained for a constant function.
- Identity Function - An identity function is a function where each element in a set B gives the image of itself as the same element i.e., g (b) = b ∀ b ∈ B. Thus, it is of the form g(x) = x.
- Algebraic functions are based on the degree of the algebraic expression. The important algebraic functions are linear function, objective functions, quadratic function, cubic function, polynomial function.

9.5. Ordered Pairs

As the name implies, an ordered pair is a set of items where the order in which they are placed is of particular significance. In coordinate geometry, ordered pairs are typically used to represent a point on a coordinate plane. They may also be used to depict aspects of a relationship. When two items are written inside parentheses and are separated by a comma, they create an ordered pair. For instance, the ordered pair (x, y) indicates an ordered pair in which 'x' is referred to as the first element and 'y' is referred to as the second element. These items, which can be either variables or constants, have distinct names depending on the context in which they are used. In an ordered pair, the element order is quite significant. It implies that (x, y) may not always be
equivalent
to (y, x). (2, 8), (a, b), (6, -5), etc are some examples of ordered pairs.

First Element of Ordered Pair	Second Element of Ordered Pair
It is called x-coordinate.	It is called y-coordinate.
Another name for this is "abscissa".	Another name for this is "ordinate".
It represents the horizontal distance of the point from the origin.	It represents the vertical distance of the point from the origin.
This number is one of the numbers on the x-axis.	This number is one of the numbers on the y-axis.
It represents the distance of the point from the y-axis.	It represents the distance of the point from the x-axis.

9.5.1. Ordered Pair in Coordinate Geometry

In coordinate geometry an ordered pair is used to represent the position of a point on the coordinate plane with respect to the origin. A coordinate plane is formed by two perpendicular intersecting lines among which one is horizontal (x-axis) and the other line is vertical (y-axis).

The intersecting point of both axes is the origin. Every point on the coordinate plane is represented by an ordered pair (x, y) where the first element x is called the x-coordinate and the second element y is called the y-coordinate.

We can see more differences between the elements of the ordered pair used in geometry here.

9.5.2. Graphing Ordered Pairs

We now know the distinction between an ordered pair's x- and y-coordinates in coordinate geometry. Now let's look at how to graph ordered pairs.

Step 1: Always begin at the origin and travel |x| units horizontally to the right for positive x and to the left for negative x. Remain there.

Step 2: Begin where you left off in Step 1 and travel vertically by |y| units either up or down depending on the value of y. Remain there.

Step 3: Exactly where you ended in Step 2 should be marked with a dot, which indicates the ordered pair (x, y).

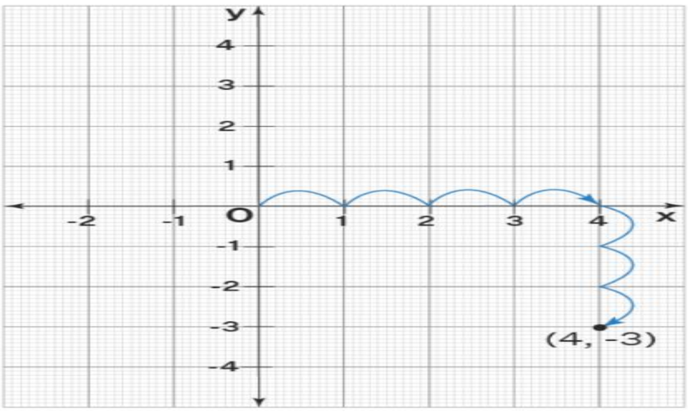
In these steps, |x| and |y| represent the absolute values of x and y respectively.

9.5.3. Ordered Pairs in Different Quadrants

Example: Graph the ordered pair (4, -3).

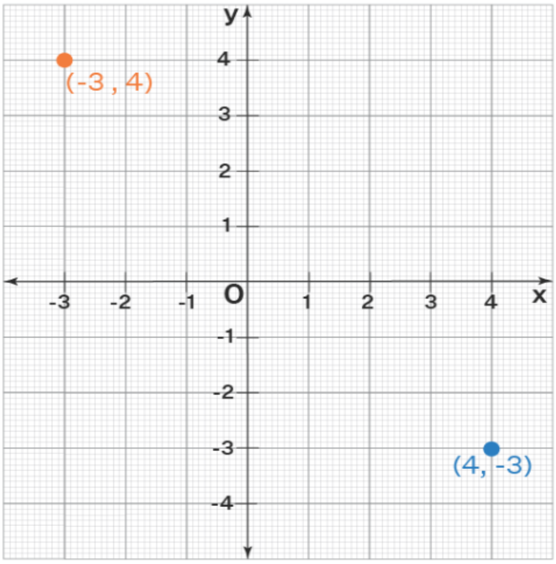
Let's start from the origin, move to the right by 4 units (as 4 is positive) and then move down by 3 units (as 3 is negative)

Graphing (4,-3)



The order of the elements in an ordered pair is important and hence the name "ordered" pair. For example, (4, -3) and (-3, 4) are located at different positions on the plane as shown below.

Graphing (4, -3) and (-3, 4)



The x and y axes in the image above divide the coordinate plane into four equal pieces. Each of these four pieces is referred to as a quadrant. According to the quadrant, the signs of x and y in an ordered pair (x, y) of a point vary, and they are displayed in the table below.

Quadrant	Ordered Pair Signs
I	$x > 0, y > 0$
II	$x < 0, y > 0$
III	$x < 0, y < 0$
IV	$x > 0, y < 0$

For example, (2, -4) refers to 2 on the x-axis (positive) and -4 on the y-axis (negative). So (2, -4) is a point in quadrant IV.

9.5.4. Ordered Pair in Sets

We have seen how coordinate geometry locates a point by using ordered pairs. But they also have a distinct application in set theory. The cartesian product is the collection of all feasible ordered pairings from set A to set B. It is a set made up of all ordered pairs (x, y) where x is in A and y is in B. For instance, if A = 1, 2, 3, and B = a, b, c, then the cartesian product is A x B = (1, a), (1, b), (1, c), (2, a), (2, b), (2, c), and (3, a), (3, b), (3, c). A relation is any subset of the cartesian product. For instance, "(1, a), (b), (c)" is a relation

9.5.5. Equality Property of Ordered Pairs

If (x, y) = (a, b), then x = a and y = b for any two ordered pairs (x, y) and (a, b) (either in coordinate geometry or in relations). In other words, if two ordered pairs have identical values, then so do their corresponding elements. The "equality property of ordered pairs" refers to this. For instance:

X = 2 and Y = -3 if (x, y) = (2, -3).

9.6. Summary

- ✔ Relation R from set A to set B is a subset of $A \times B$
- ✔ R is a relation on set A and if $(x,x) \in R$ for all $x \in A$ then R is reflexive
- ✔ R is a relation on set A and if $(x,y) \in R$ and $(y,x) \in R$ for all $x ,y \in A$ then R is symmetric.
- ✔ R is a relation on set A and if $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$ for all $x ,y,z \in A$ then R is transitive.
- ✔ If R is reflexive, symmetric and transitive then R is an equivalence relation.
- ✔ If $f: A \rightarrow B$ is a function and f(x) = y where $x \in A$ and $y \in B$
- ✔ In ordered pair (a,b) a is called first and b is second component and (a,b) =(x,y) then a =x and b = y

9.7. Self-Assessment questions

1. Which relations '>' in the set of N is

   A) Symmetric B) Reflexive C) Transitive D) Equivalence relation.
2. If (1/x ,1/y) = (15.2), Find the values of x and y.
3. Express $\{(x,y) / x^2 + y^2 = 100$, where x,y € W} as a set of ordered pairs.
4. Check R: $Z \rightarrow \mathcal{Z}$, R = {(a , b) / 2 divides a-b} is an equivalence relation.
5. If $f(x) = 2x^2 + bx + c$, f(0) = 3, f(2) = 1 then find f(1) =
6. Check if the following relation is a function R = {(p,a), (p,b),(q,c) ,(r,d),}
7. Let the function f be defined by f(x) = (2x + 1)/(1-3x) find f(2) and f(-1)
8. How to determine Relation is a function?
9. Why all functions arerelated but all relation is not function.
10. What are the domains and the range of the function?

x	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**
y	**-3**	**0**	**3**	**6**	**9**	**12**	**15**	**18**

9.8. Reference

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Unit 10 Limits and Continuity
Learning Objectives

After studying the unit, students will be able to:

- Concept of limit and continuity
- Concept of continuity
- Various definition of continuity
- Theorems on continuity

Structure

10.1. Introduction

Limits: Your group of pals and you decide to meet somewhere outside. Is it necessary for all of your pals to live in the same area and travel there using the same route? Yes, but not always. To meet at that one location, all of the pals travel there from various locations throughout the city or country.

It appears as though various components have come together to form a single point. Mathematically, it is analogous to a function convergent to a given value. It is an illustration of boundaries. Limits demonstrate the boundedness of some functions. When its limit gets close to a certain value, the function tends toward that number.

10.2. Limits

meaning of $\chi \rightarrow a$: (χ tends to a)

1. $\chi \neq a$
2. χ assumes values nearer and nearer to a
3. χ may be slightly less than a or slightly greater than a.

Definition of limit:

$$\lim_{\chi \to a} f(\chi) = L$$

1) Let f(x) be a function of x if as x assumes values nearer and nearer to a , f(x) assumes values nearer and nearer to l, then we say f(x) tends to limit l as x tends to a.

2) if for every $\in >0$ there exist $\partial>0$ such that

$| f(x)-l | < \in$ whenever $0< | x-a| <\partial$.

The expression

$$f(x) = L$$

means that f(x) can be as close to L as desired by making x sufficiently close to 'C'. In such a case, we say that the limit of f, as x approaches to C, is L.

Limits of function: Let f be a function defined in a domain which we take to be an interval, say, I. We shall study the concept of the limitof f at a point 'a' in I.

We say lim f(x) is the expected value of f at x = a given the values of f near the x→a- left of a. This value is called the left-hand limit of f at a.

We say lim f(x) is the expected value of f at x = a given the values of f near to the X→a+ right of a. This value is called the right-hand limit of f at a.

Let f and g be two functions such that both exist, then

[f(x) ± g(x)]

1. lemlemlim = f(x) ± g(x)

$x \to a$ $x \to a$ $x \to a$

2. lemlemlim = f(x) g(x)

f(x) * g(x)

$x \to a$ $x \to a$ $x \to a$

10.3. Examples

1.Evaluate $\lim_{x\to 2} (x^2 + 5)/(x + 10)$

Solution:

We first put x=2 in $(x^2+5)/(x+10)$,

$\therefore \lim_{x\to 2} (x^2+5)/(x+10)$

$= (2^2+1)/(10+2)$

$= 5/12$

2. Evaluate $\lim_{x\to 0} (\sin x+3)^2$

Solution:

We first put x=0 in $(\sin x+3)^2$

$\therefore \lim_{x\to 0} (\sin x+3)^2$

$= (0+3)^2$

$= 9$

3.Evaluate $\lim x\to 2 \ (x^3 + x^2 + 1)$

Solution:

We first put x=1 in $x^3 + x^2 + 1$

$\therefore \lim_{x\to 2}(x^3 + x^2 +1) =8 + 4 + 1$

$\therefore \lim_{x\to 1} (x^3 - x^2+1) = 13$

4. $\lim_{x\to 2} (x^2 - 5x + 6)/(x^2 - 2x)$

Solution:

$\therefore \lim_{x\to 2} (x^2 - 5x + 6)/(x^2 - 2x)$

$= (x-3)(x-2) / x(x-2)$ cancel factor (x-2)

$= \lim_{x\to 2} (x-3) / x$

$= 2-3 /2$

$= -1/2$

10.4. Continuity

1.A function f(x) is said to be continuous at x=a

If $\lim_{x\to a} f(x)=f(a)$

2.A function is said to be continuous on the interval [a,b]

if it is continuous at each point in the interval.

3.If f(x) is continuous at x=a

then, $\lim_{x\to a} f(x)= f(a)$ $\lim_{x\to a^-} f(x) = f(a)$ $\lim_{x\to a^+} f(x) = f(a)$

Continuity And Differentiability are complementary to a function. For a function y = f(x), defined over a closed interval [a, b] and differentiable across the interval (a,b), there exists a point 'c' in the interval [a, b], such that it is continuous at the point x = c,

if $\lim_{x\to c} f(x)=f(c)$, and it is differentiable at the same point x = c,
if $\lim_{x\to c} f'(c)=f(c+h)-f(c)h/ h$.

Additionally, before the function is differentiable at a point, its continuity at the point must first be established. Let's find out more about the rules, proofs, and illustrations of continuity and differentiability.

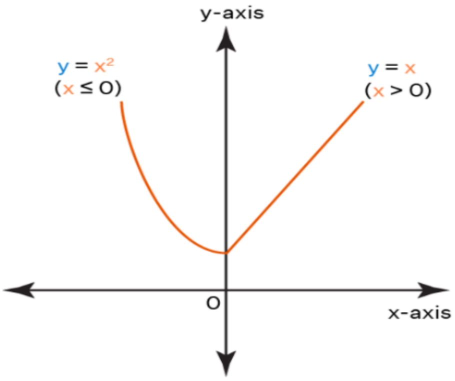
Both a function's continuity and differentiability are beneficial to one another. Prior to being proven for its differentiability at the point x = a, the function y = f(x) must first be proven for its continuity at the point x = a. Both geometrically and algebraically, the ideas of continuity and differentiability may be shown.

The continuity of a function f(x) at the point x = c can be proved if the limit of the function at the point is equal to the value of the function at the same point. $\lim_{x\to c} f(x)=f(c)$. The derivative of a function y = f(x) is defined as f'(x) or d/dx.f(x) and is represented as $f'(x)=\lim_{x\to c} f(x+h)-f(x)/h$.

10.4.1. Continuity Of A Function

If we can easily draw the graph without lifting the pencil at a point, then the graph y = f(x) is said to be continuous. Let c be a point residing in the function's domain and f(x) be a real-valued function on the subset of real numbers (x). The function f(x) is said to be continuous at the point x = c if we have

Limx→cf(x)=f(c)



A function's continuation can be demonstrated visually or algebraically. A graph line that continually and uninterruptedly passes through a point on a function, such as y = f(x), is said to have continuity. If the value of the function from the left-hand limit equals the value of the function from the right-hand limit, the continuity of a function y = f(x) may be observed algebraically.

Limx→1−1f(x)=Limx→1+1f(x). That is, the values of x = 0.99, 0.998, slightly lesser than 1, have the same f(x) function value as that for x = 1.001, 1.0001, which are slightly greater than 1.

10.5. Theorems on Continuity

The essential continuity and differentiability theorems listed below provide the necessary context for a greater comprehension of these ideas.

Theorem 1 states that if two functions, f(x) and g(x), are continuous at both a real valued function and a point called x = c, then the following is true:

At x = c, f(x) + g(x) is continuous.

At x = c, f(x) - g(x) is continuous.

At x = c, g(x).g(x) is continuous.

f(x)/g(x) is continuous at a point x = c, provided $g(c) \neq 0$

Theorem2 : states that the composite function fog(x) is defined at x = c for two real value functions f(x) and g(x). If the function f(x) is continuous at g(c) and g(x) is continuous, then fog(x) is continuous at x = c.

Theorem 3: A function f(x) is continuous at a position x = c if and only if it is differentiable at that location. Every differentiable function is continuous, to put it simply.

Chain Rule Theorem 4: For a real valued function f(x), which is a combination of two functions u and v ie, f = vou Let's assume that dt/dx and dv/dt exist if t = u(x) and that df/dx = dv/dt.dt.dx.

Theorem 5: The derivative of e^x with respect to x is e^x. d/dx.e^x = 1. And the derivative of log x with respect to x is 1/d. d/dx. log x = 1/x.

Theorem 6: (Rolle's Theorem). If a function f(x) is continuous across the interval [a, b] and differentiable across the interval (a, b), such that f(a) = f(b), and a, b are some real numbers. Then there exists a point c in the interval [a, b] such that f'(c) = 0.

Theorem 7: (Mean Value Theorem). If a function f(x) is continuous across the interval [a, b] and differentiable across the interval (a, b), then there exists a point c in the interval [a, b] such that f′(c)=f(b)−f(a)/b−a.

10.6. Summary

- ✔ Continuity of a Function is feasible to draw the curve without encountering any breaks, the function is considered to be continuous in the interval. If every point in the interval satisfies the function, the function is said to be continuous. If and only if, a function is continuous when x = a.
- ✔ Non-Linear Equations that is not linear in the unknown function and its derivatives is referred to as a non-linear differential equation
- ✔ Continuous function is If a function f(x) meets the requirements below, then it is said to be continuous in the closed interval [a,b].
- ✔ 1) The open interval shows that f(x) is continuous (a, b)
- ✔ 2) At the position a from the right, f(x) is continuous,
  i.e. lim x → a f (x) =f(a)

10.7. Self-assessment Questions

1. What are the limits of a function?
2. What is left hand and right-hand limit
3. Define continuity of a function.
4. Evaluate $\frac{\tan\ 5x}{x}$
5. Evaluate $\frac{x+3}{x^2+4x+3}$
6. *Evaluate* $\frac{x+3}{x-5}$
7. Write any two theorems on continuity
8. Explain various definitions of continuity.

9. Evaluate $\frac{x-2}{x^2-6x+8}$

10. Evaluate $\frac{x-5}{x^2-8x+15}$

10.8. Reference

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Unit 11

Derivatives

Learning Objectives

After studying the unit, students will be able to:

- Geometrical meaning of Derivative
- Several formulas of different function
- Rules of Derivatives
- Composite function and implicit function
- Higher order derivative

Structure

11.1. Introduction

11.2. Derivative of a function using first principle

11.3. Derivative Formulae

11.4. Chain Rule

11.5. Implicit Functions

11.6. Parametric Function

11.7. Higher order Derivative

11.8. Summary

11.9. Self-assessment Questions

11.10. Reference

11.1. Introduction

- **Derivative as Slope of Curve**

To find the slope of a curve at a given point, we simply differentiate the equation of the curve and find the first derivative of the curve, i.e., dy/dx.

Let's find out the answer with an example.

Explanation:

To determine slope, we use the following steps:

- Firstly, we need to differentiate the given equation or simply we have to find the dy/dx of an equation.
- After differentiating we will get the equation of slope.
- Put the value of x in the equation to determine the slope.

Let's take an example to find the slope of a curve at a given point.

Example: Determine the slope of the curve $y = x^4 – x^3 + 1$ at the given point (2,−15).

Given,

Equation ⇒ $y = x^4 – x^3 + 1$,

Point = (2,−19)

First we need to differentiate the given equation ($y = x^4 – x^3 + 1$),

$dy/dx = d(x^4 – x^3 + 1)/dx$

$dy/dx = d(x^4)/dx − d(x^3)/dx + d(1)/dx$

$dy/dx = 4x^3 – 3x$

Now, put the value of (x = 2) in the equation to determine the slope.

$dy/dx = 4(2)^3 – 3(2)$

$dy/dx = 26$

Thus, 26 is the slope of the given curve.

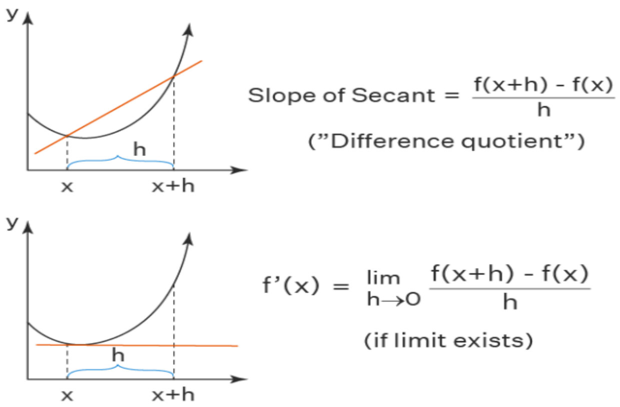
Hence, to find the slope of a curve at a given point, we simply differentiate the equation of the curve and find the first derivative of the curve, i.e., dy/dx.

Geometrical Interpretation

Calculus defines a derivative as the rate at which one quantity, y, changes in relation to another, x. The differential coefficient of y with respect to x is another name for it. Finding a function's derivative is the process of differentiation.

A function's derivative is often denoted by d/dx (f(x)) (or df/dx (or Df(x) (or f')) (x). Let's examine the technical definition of a derivative. Two points on a function f(x) curve are (x, f(x)) and ((x + h), f(x + h)), respectively. If so, [f(x + h) - f(x)]/(x + h - x) = [f(x + h) - f(x) / h] will be the slope of the secant line across these locations. The second point overlaps the first point in the image below when the distance between two points is almost equal to 0 (i.e., when h approaches 0), and the secant line turns into the tangent line. Calculus refers to the slope of the tangent line as the function's derivative. i.e.,

- The derivative of the function, f '(x) = Slope of the tangent = $\lim_{h\to 0}$ [f(x + h) - f(x) / h.



This formula is popularly known as the "limit definition of the derivative" (or) "derivative by using the first principle".

In mathematics, the derivative of a function f(x) is represented by the symbol f'(x), which has the following contextual interpretation:

- The slope of the tangent drawn to the curve at a given location is the derivative of a function at that point.
- The instantaneous rate of change at a particular point on the function is likewise represented by it.
- The derivative of the displacement function can be used to determine a particle's velocity.
- A function is optimised (maximized/minimized) using the derivatives.
- The intervals where the function is rising or decreasing as well as the intervals where the function is concave up or down are also found using them.

Thus, whenever we see the phrases like "slope/gradient", "rate of change", "velocity (given the displacement)", "maximize/minimize" etc then it means that the concept of derivatives is involved.

11.2. Derivative of a Function Using the First Principle

The derivative of a function can be obtained by the limit definition of derivative which is f'(x) = $\lim_{h\to 0}$ [f(x + h) - f(x) / h. This process is known as the differentiation by the first principle.

Let $f(x) = x^2$ and we will find its derivative using the above derivative formula. Here, $f(x + h) = (x + h)^2$ as we have $f(x) = x^2$. Then the derivative of f(x) is,

$f'(x) = \lim_{h\to 0} [(x + h)^2 - x^2] / h$

$= \lim_{h\to 0} [x^2 + 2xh + h^2 - x^2] / h$

$= \lim_{h\to 0} [2xh + h^2] / h$

$= \lim_{h\to 0} [h(2x + h)] / h$

$= \lim_{h\to 0} (2x + h)$

$= 2x + 0$

$= 2x$

So, 2x is the derivative of x2. Finding the derivatives of complicated functions using this limit approach, however, could be challenging. As a result, there are a few derivative formulae (naturally generated from the above limit definition) that we may apply easily during the differentiation process.

11.3. Derivative Formula

The first principle of differentiation is used to determine the three fundamental derivatives of the algebraic, logarithmic/exponential, and trigonometric functions, which are then used as the basis for common derivative formulae. They are listed below.

11.3.1. Power Rule of Derivatives

By using the above example, the derivative of x^2 is 2x. Similarly, we can prove that the derivative of x^3 is $3x^2$, the derivative of x^4 is $4x^3$, and so on. Power rule generalizes this and it is stated as $d/dx\ (x^n) = n\ x^{n-1}$.

11.3.2. Derivatives of Algebraic function

Here are the derivatives of Algebraic functions.

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{d_x}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

11.3.3. Derivatives of log/Exponential Functions

Here are the derivatives of trigonometric functions.

$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$\frac{d}{dx}\left(a^x\right) = a^x \log a$$

$$\frac{d}{dx}\left(\log x\right) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\log_a^x\right) = \frac{1}{x.\log a}$$

11.3.4. Fundamental Rules of Derivatives

The following are the fundamental rules of derivatives. Let us discuss them in detail.

Power Rule: By this rule, if y = x^n , then dy/dx = n x^{n-1} . Example: d/dx (x^5) = $5x^4$.

Sum/Difference Rule: The derivative process can be distributed over addition/subtraction. i.e., $\frac{d}{dx}\left(u \pm v\right) = \frac{du}{dx} \pm \frac{dv}{dx}$

Product Rule: According to the product rule of derivatives, a function's derivative is the derivative of the second function multiplied by the first function plus the derivative of the first function multiplied by the second function if the function is a product of two functions. dy/dx [u × v] = u · dv/dx + v · du/dx.

$$\frac{d}{dx}\left(uv\right) = u\frac{dv}{dx} + v\frac{du}{dx}$$

If y = x^5 e^x , we have y' = x^5 . e^x + e^x . $5x^4$ = e^x (x^5 + $5x^4$)

Quotient Rule: The quotient rule of derivatives states that

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2} \quad (v \neq 0)$$

Constant multiple Rule: The constant multiple rule of derivatives states that d/dx [c(f(x)] = c · d/dx f(x). i.e., the constant which when multiplied by a function, comes out of the differentiation process. For example, d/dx ($5x^2$) = 5 d/dx (x^2) = 5(2x) = 10 x.

Constant Rule: The constant rule of derivatives states that the derivative of any constant is 0. If y = k, where k is a constant, then dy/dx = 0. Suppose y = 4, y' = 0. This rule directly follows from the power rule.

11.4. Derivatives of Composite Functions (Chain Rule)

F(g(x)) is also differentiable if f and g are both differentiable functions in their domain. For composite functions, this is known as the chain rule of differentiation.

(fog)'(x) = f'[(g(x)] g'(x). This also can be written as "if y = f(u) and u = g(x) , then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

For example, consider y = $(x^2 -5x +2)^3$. This is a composite function. We can write this function as y = u^3, where u = x^2 -5x +2. Then

dy/du = $3u^2$

du/dx = d/dx (x^2 -5x +2) = 2x -5

By the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot (2x -5)$

$= 3 (x^2 -5x +2)^2 (2x -5)$

11.5. Derivatives of Implicit Functions

Implicit differentiation is used in equations where the variables x and y cannot clearly specify y as a function of x. If f(x, y) = 0, then group the terms containing dy/dx at one side and differentiate on both sides with regard to x before solving for dy/dx.

For example, 4x + y = 10

d/dx(4x + y) = d/dx(0)

4 + dy/dx = 0

dy/dx = -4

11.6. Parametric Function Derivatives

In a function, we may have the dependent variables x and y which are dependent on the third independent variable. If x = f(t) and y = g(t), then the derivative is calculated as dy/dx = f'(x)/g'(x). Suppose, if x = 5 + t^2 and y = $6t^2 -7t^4$, then we find dy/dx as follows.

$$\frac{dy}{dx} = \frac{dy\ /dt}{dx\ /dt}$$

dx/ dt = 2t and dy/dt = 12t -$28t^3$

dy/dx = (dy/dt)/(dx/dt)

dy/dx = (12t -$28t^3$)/2t

$= 2t\,(6 - 19t^2)\,/\,2t$

$dy/dx = 6 - 19t^2$

11.7. Higher-order Derivatives

We can find the successive derivatives of a function and obtain the higher-order derivatives. If y is a function, then its first derivative is dy/dx. The second derivative is d/dx (dy/dx) which also can be written as d^2y/dx^2. The third derivative is d/dx (d^2y/dx^2) and is denoted by d^3y/dx^3 and so on.

Alternatively, the first, second, and third derivatives of f(x) can be written as f'(x), f''(x), and f'''(x). For higher order derivatives, we write the number in brackets as the exponent. Suppose $y = 9x^3$, we get the successive derivatives as follows. $y' = 27x^2$, $y'' = 54\,x$ and $y''' = 54$, $y^{(4)} = 0$.

11.8. Summary

- ✔ Derivative means slope of tangent to the given curve
- ✔ Power Rule: By this rule, if $y = x^n$, then $dy/dx = n\,x^{n-1}$.
- ✔ Sum/Difference Rule: The derivative process can be distributed over addition/subtraction. i.e., $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$
- ✔ Product Rule: According to the product rule of derivatives, a function's derivative is the derivative of the second function multiplied by the first function plus the derivative of the first function multiplied by the second function if the function is a product of two functions. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
- ✔ Quotient Rule: The quotient rule of derivatives states that
- ✔ $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$ $(v \neq 0)$
- ✔ Constant multiple Rule: The constant multiple rule of derivatives states that d/dx [c(f(x)] = c · d/dx f(x). i.e., the constant which when multiplied by a function, comes out of the differentiation process
- ✔ Explicit function is y =f(x) and implicit function is f(x,y)=0
- ✔ The second derivative is d/dx (dy/dx) which also can be written as d^2y/dx^2

11.9. Self-Assessment Questions

1. $\sqrt{xy}$ =1 then find $\frac{dy}{dx}$

2. Differentiate following w.r.t x $(5x^4 - 7x^2 + 3x - 5)^4$
3. Differentiate following w.r.t x (x-3)(x-4)
4. Differentiate following w.r.t x $(x^3 - 5x + 7) + (2x^4 - 3x + 5)$
5. Write formula for Derivative of product
6. Write formula for Quotient rule for Derivative
7. Explain explicit and implicit functions of derivatives.
8. Find derivative of log (2+x)
9. Find derivative of $x^2 - xy + y^2 - 7 = 0$
10. Find second order derivative of $y = x^3 - 7x^2 - 5x + 5$

11.10. Reference

- Apostol, T. (1991): Calculus, Volumes 1 and 2, Wiley.
- Simmons, G. and Krantz, S. (2006): Differential Equations, McGraw Hill.
- https://www.tiwariacademy.com/ncert-solutions/class-12/maths/Unit-6/

Unit 12

Integration

Learning Objectives

After studying the unit, students will be able to:

- Geometrical meaning of integration
- Meaning of antiderivative
- Various formulae of integration

Structure

12.1. Introduction

12.2. Integration Definition

12.3. Definite Integral

12.4. Rules of integration

12.5. Properties of Integration

12.6. Integration using Partial Fractions

12.7. Summary

12.8. Self-assessment questions

12.9. Reference.

12.1. Introduction

Calculating an integral is called integration. Mathematicians utilize integrals to determine a variety of useful quantities, including areas, volumes, displacement, etc. When we discuss integrals, we typically refer to definite integrals. Anti derivatives are calculated using indefinite integrals. Apart from differentiation, integration is one of the two main calculus subjects in mathematics (which measure the rate of change of any function with respect to its variables).. You will discover the definition of integrals in mathematics, the integration formulas, and examples here.

12.2. Integration Definition

The summing of discrete data is indicated by the integration. To determine the functions that will characterise the area, displacement, and volume that result from a combination of small data that cannot be measured separately, integrals are calculated. The concept of limit is employed broadly in calculus whenever algebra and geometry are applied. Limits assist us in analysing how points on a graph behave, such as how they approach one another until their distance approaches zero. There are two main categories of calculus that we are aware of:

- Differential Calculus
- Integral Calculus

The idea of integration has evolved to address the following issues:

- When a function's derivatives are known, to locate the problematic function.
- To determine the region constrained by the graph of a function.

Due to these two issues, the "Integral Calculus," which consists of definite and indefinite integrals, was developed. The Fundamental Theorem of Calculus connects the ideas of differentiating a function and integrating a function in calculus..

In mathematics, integration is a technique for combining or adding the parts to arrive at the total. It is a form of differentiation in reverse where we break down functions into their component elements. This technique is employed to determine the summation on a sizable scale. Small addition problem calculation is a simple task that can be completed manually or with the aid of a calculator. Integration techniques are employed for large addition problems, when the bounds potentially extend to infinity. Calculus includes both integration and differentiation, both of which are crucial. These subjects have a very high conceptual level. As a result, it is first exposed to us in upper secondary classrooms before moving on to engineering or further education. Read the entire article to have a thorough understanding of integrals.

Integration – Inverse Process of Differentiation

We are aware that differentiation is the process of discovering a function's derivative and integration is the process of discovering a function's antiderivative. Thus, both processes are the antithesis of one another. Therefore, we can say that differentiation is the process of differentiation and integration is the reverse. The anti-differentiation is another name for integration. In this procedure, we are given a function's derivative and asked to determine the function (i.e., primitive).

We know that the differentiation of sin x is cos x.

It is mathematically written as:

$(d/dx) \sin x = \cos x$ …(1)

Here, cos x is the derivative of sin x. So, sin x is the antiderivative of the function cos x. Also, any real number "C" is considered as a constant function and the derivative of the constant function is zero.

So, equation (1) can be written as

$(d/dx) (\sin x + C)= \cos x +0$

$(d/dx) (\sin x + C)= \cos x$

Where "C" is the arbitrary constant or constant of integration.

Generally, we can write the function as follow:

$(d/dx) [F(x)+C] = f(x)$, where x belongs to the interval I.

To represent the antiderivative of "f", the integral symbol "∫" symbol is introduced. The antiderivative of the function is represented as $\int f(x)\, dx$. This can also be read as the indefinite integral of the function "f" with respect to x.

Therefore, the symbolic representation of the antiderivative of a function (Integration) is:

$y = \int f(x)\, dx$

$\int f(x)\, dx = F(x) + C$.

Indefinite Integral

Indefinite integrals are defined without upper and lower limits. It is represented as:

$\int f(x)dx = F(x) + C$

Where C is any constant and the function f(x) is called the integrand.

12.3. Definite Integral

A definite integral is one that has both the upper and lower bounds. x can only be allowed to lay on an actual line. The Definite Integral is also known as a Riemann Integral.

A specific Integral is shown as:

$$\int_a^b f(x) = [\phi(x)]_a^b$$
$$= \phi(b) - \phi(a)$$

12.4. Rules of integration

1. Sum and Difference Rules:

- $\int [f(x)+g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$
- $\int [f(x)-g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$

2. Power Rule: $\int x^n dx = (x^{n+1})/ (n+1)+ C$. (Where $n \neq -1$)

3. Exponential Rules:

- $\int e^x\, dx = e^x + C$
- $\int a^x dx = a^x/\ln(a) + C$
- $\int \ln(x)dx = x \ln(x) - x + C$

4.Constant Multiplication Rule:

- $\int a\, dx = ax + C$, where a is the constant.

5.Reciprocal Rule:

- $\int (1/x)\, dx = \ln(x)+ C$

12.5. Properties of Integration

A few properties of indefinite integrals are:

- $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
- $\int k\, f(x)\, dx = k \int f(x)\, dx$, where k is any real number.
- $\int f(x)\, dx = \int g(x)\, dx$, if $\int [f(x)-g(x)]\, dx = 0$
- The combination of first two properties arise to
  $\int[k1f1(x)dx+k2f2(x)dx+.........knf(x)dx]=k1\int f1(x)dx+k2\int f2(x)dx+....+kn\int f(x)dx$

Examples

1. Integrate $(2x^3 - x^2+1)/x^4\, dx$,

Solution:

$\int (2x^3 - x^2 +1)/x^4\, dx = \int (2x^3 /x^4 - x /x+1/x^4)$

$= \int (1/x)dx - \int (1/x^2)\, dx + \int (1/x^4)dx$

Applying the reciprocal rule and the power rule, we get

$\int (2x^3 – x^2 +1)/x^4 dx = \log|x| + 1/x - 1/3x^3 + C$

2: Find the integral of the function: $\int x^3 dx$

Solution:

Given $\int x^3 dx$

$= (x^4/4) + C.$

3. Integrate $\int (x^3-1)(5-3x)dx$.

Solution:

Given: $\int (x^3-1)(5 - 3x)dx$.

Multiply the terms, we get

$\int (x^3-1)(5- 3x)dx = \int (5x^3+3x^4 + 3x – 5) dx$

Now, integrate it, we get

$\int (x^3-1)(5- 3x)dx = 5(x^4/4) + 3(x^5/5)- 3(x^2/2) – 5x + C$

4. If $d/dx[f(x)] =3x^3−2x^4$ and $f(1)=0$**,** find the function f(x)

Solution:

Integration on both sides of the equation $d/dx[f(x)] = 3x^3–2x^4$

$\int d/dx[f(x)]dx=\int(3x^3–2x^4)dx$

$f(x)=3(x^4/4) − 2x^5/5 + C$

$f(x)= 3(x^4/4) – 2(x^5/5) +C$

Use the condition f(1)=0 after integration, to find the value of C.

$f(1) = 3 – 2 + C = 0$

$C = -1$

Answer: ∴ the function is $f(x)= 3(x^4/4) – 2(x^5/5) -1$

12.6. Integration using Partial Fractions

Suppose we have to find $y=\int P(x)/Q(x)dx$

where P(x)/Q(x) is an improper rational function. We reduce it in such a way that P(x)/Q(x)=T(x)+ P1(x)/Q(x). Here, T(x) is polynomial in x and P1(x)Q(x) is proper rational function. .

Example

Integrate $f(x)=1/(x- 1)(x-3)$

Solution:

By using partial fraction we have $1/(x-1)(x-3)=A/x-1+ B/ x-3$ ⋯(1)

We will determine the values of A and B.

On comparing in equation (1), we get 1=A(x-3)+B(x-1).

From this, put x=1

1 = A(-2)

A = -1/2

Put x=3

1=B(2)

B =1/2

So, equation (1) can be written as 1/(x-1)(x-3)= - 1/2(x-1)+1/2(x-3)

Now, solving the integral

∫(1(x+1)(x+2))dx=∫(- 1/2(x-1)+1/2(x-3) dx

= - ½ log |x-1|+ ½ log |x- 3 |+ C

12.7. Summary

- ✔ $\int x^n dx = (x^{n+1})/(n+1) + C$. (Where $n \neq -1$)
- ✔ $\int e^x dx = e^x + C$
- ✔ $\int a^x dx = a^x/\ln(a) + C$
- ✔ $\int \ln(x) dx = x \ln(x) - x + C$
- ✔ $\int a\, dx = ax + C$, where a is the constant.
- ✔ $\int (1/x)\, dx = \ln(x) + C$

12.8. Self-assessment questions

I. Find the following integrals.

1. $\int (5x^2 - 8x + 5)dx$

2. $\int (-6x^3 + 9x^2 + 4x - 3)dx$

3. $\int (x^{\frac{3}{2}} + 2x + 3)dx$

4. $\int \left(\frac{8}{x} - \frac{5}{x^2} + \frac{6}{x^3} \right) dx$

5. $\int (\sqrt{x} + \frac{1}{3\sqrt{x}})dx$

6. $\int (12x^{\frac{3}{4}} - 9x^{\frac{5}{3}})dx$

7. $\int \frac{x^2 + 4}{x^2} dx$

8. $\int \frac{1}{x\sqrt{x}} dx$

9. $\int (1 + 3t)t^2 \, dt$

10. $\int (2t^2 - 1)^2 \, dt$

12.8. Reference:

- ✔ *https://www.cuemath.com/calculus/integration/*
- ✔ https://byjus.com/maths/integration/

Unit 13

Maxima and Minima

Learning Outcomes:

After studying the unit, students will be able to:

Structure

13.1. Introduction

13.2. Maxima and Minima of a Function

13.3. Working procedure of Maxima and Minima

13.4. Properties of maxima and minima

13.5. Examples

13.6. Summary

13.7. Self-assessment questions

13.8. Reference

13.1. Introduction

The extrema of a function are the maxima and minima. The maximum and minimum values of a function inside the specified ranges are known as maxima and minima, respectively. Absolute maxima and absolute minima are terms used to describe the function's maximum and minimum values, respectively, over its full range.

The terms "local maxima" and "local minima" refer to additional maxima and minima of a function that are not its absolute maxima and minima. Learn more about finding the function's maximum and minima, as well as local and absolute maxima and minima.

13.2. Maxima and Minima of a Function

The curve of a function has peaks and troughs called maxima and minima. A function may have any number of maxima and minima. Calculus allows us to determine any function's maximum and lowest values without ever consulting the function's graph. Maxima will be the curve's highest point within the specified range, and minima will be its lowest.

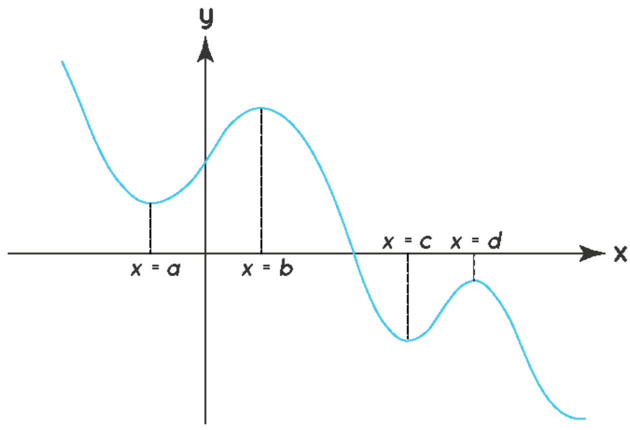
Extrema is the result of maxima and minima combined. The graph in the graphic below shows several peaks and falls. We obtain the function's maximum and minimum values at $x = a$ and 0 respectively, and at $x = b$ and c respectively.

13.2.1. Local Maxima and Minima

The maxima and minima of the function that appear in a certain interval are known as local maxima and minima. The value of a function at a place in a certain interval for which the values of the function near that point are always less than the value of the function at that point would be considered a local maxima. Local minima, on the other hand, would be the value of the function at a location where the values of the function nearby are higher than the value of the function at that location

Local Maxima: A point $x = b$ is a point of local maximum for $f(x)$ if in the neighborhood of b i.e in $(b-\square, b+\square)$ where □ can be made arbitrarily small, $f(x) < f(b)$ for all $x \in (b-\square, b+\square) \setminus \{b\}$. This simply means that if we consider a small region (interval) around $x = b$, $f(b)$ should be the maximum in that interval.

Local Minima: A point $x = a$ is a point of local minimum for $f(x)$ if in the neighbourhood of *a*, i.e. in $(a-\square,a+\square)$, (where □ can have arbitrarily small values), $f(x) > f(a)$ for all $x \in (a-\square,a+\square) \setminus \{a\}$. This means that if we consider a small interval around $x = a, f(a)$ should be the minimum in that interval.

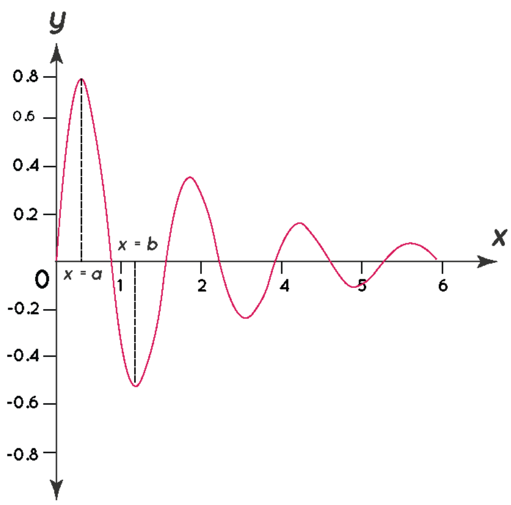


13.2.2. Absolute Maxima and Minima

Absolute maxima of a function refers to its highest point within its whole domain, whereas an absolute minimum of a function refers to its lowest point within its entire domain. Over the whole domain, a function can only have one absolute maximum and one absolute minimum. The global maxima and global minima of the function are another name for the absolute maxima and minima of the function.

- Absolute maxima: A point x = a is a point of global maximum for f(x) if $f(x) \leq f(a)$ for all $x \in D$ (the domain of f(x)).
- Absolute minima: A point x = a is a point of global minimum for f(x) if $f(x) \geq f(a)$ for all $x \in D$ (the domain of f(x)).

In the image given below, point x = a is the absolute maxima of the function and at x = b is the absolute minima of the function.



13.3. Working procedure of Maxima and Minima

Both the first-order derivative test and the second-order derivative test can be used to determine a function's maximum and minimum. The quickest methods for locating a function's maxima and minima are derivative tests. Let's go over each one individually.

13.3.1. First Order Derivative Test

The slope of a function can be found in its first derivative. The slope of the curve grows as we approach a maximum point, reduces as we go away from the maximum point, and ultimately becomes zero at the maximum point. In a similar manner, as we approach the minimum point of the function, the slope of the function reduces, reaches 0 at the minimum point, and then increases as we travel away from the minimum point. This data helps us determine whether the position is a peak or minima.

Let's imagine we have a function f defined in an open interval I, which is continuous at the critical point, and $f'(c) = 0$. (slope is 0 at c). Once we have determined the nature of $f'(x)$ and its values at the curve's left and rightmost points, we may predict what the given point will be.

- Local maxima: If f'(x) changes sign from positive to negative as x increases via point c, then f(c) gives the maximum value of the function in that range.
- Local minima: If f'(x) changes sign from negative to positive as x increases via point c, then f(c) gives the minimum value of the function in that range.
- Point of inflection: If the sign of f'(x) doesn't change as x increases via c, and the point c is neither the maxima or minima of the function, then point c is called the point of inflection.

13.3.2. Second-Order Derivative Test

The first derivative of the function is determined in the second-order derivative test for maxima and minima if the slope is equal to 0 at the critical point $x = c$ ($f'(c) = 0$), in which case the second derivative of the function is determined. The supplied point will be: if the second derivative of the function occurs inside the given range.

- Local maxima**:** If $f''(c) < 0$
- Local minima**:** If $f''(c) > 0$
- Test fails:If $f''(c) = 0$

13.4. Properties of maxima and minima

1. If f(x) is a continuous function in its domain, at least one maximum or minimum must exist between two equal values of f (x).

2. Maximums and minima alternately occur. In other words, there is one maxima and two minima between them.

3. If f(x) goes to infinity when x tends to an or b and f'(x) only equals zero for one value of x, namely, c between a and b, then f(c) is the smallest and least significant value. When f(x) tends to - as x tends to an or b, the maximum and highest value is f(c).

13.5. Examples

Example 1: Examine for maxima and minima of the function $y = 2x^3 - 6x^2 - 18x + 20$.

Answer: For turning points dy/dx = 0.

$dy/dx = 6x^2 - 12x - 18 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x - 3)(x + 1) = 0$

$\Rightarrow$ x= 3 and x = (-1)

Second Derivative Test:

At x = -1 :

$d^2 y/dx^2 = 12x - 12 = 12(-1) - 12 = -24 < 0$

Hence x = -1 is point of maxima

At x = 3

$d^2y/dx^2 = 12x - 12 = 12(3) - 12 = 24 > 0$

Hence x = 3 is a point of minima.

Example 2: Find the local maxima and minima of the function

$f(x) = (¼)x^4 + (1/3)x^3 – x^2 + 12$.

Answer:

For stationary points f'(x) = 0.

$f'(x) = x^3 + x^2 – 2x = 0$

$\Rightarrow x(x^2 + x – 2) = 0$

$\Rightarrow x(x – 1)(x + 2) = 0$

$\Rightarrow$ Hence, x = 0, x = 1 and x = -2

Second derivative test:

$f''(x) = (3x^2 + 2x – 2)$

At $x = -2$

$f''(-2) = (3(-2)^2 + 2(-2) – 2) = (12 – 4 – 2) = (6) > 0$

At $x = 0$

$f''(0) = (3(0)^2 + 2(0) – 2) = (-2) = -2 < 0$

At $x = 1$

$f''(1) = (3(1)^2 + 2(1) – 2) = (3+2-2) = (3) = 3 > 0$

Therefore, by the second derivative test $x=0$ is the point of local maxima while $x = -2$ and $x=1$ are the points of local minima.

Example 3: Find the points of maxima and minima of a function = $2x^3 - 3x^2 + 6$

Solution

Given function: $y = 2x^3 - 3x^2 + 6$

Using second order derivative test for the maxima and minima of a function:

Taking first order derivative of:

$y = 2x^3 - 3x^2 + 6$ ----- (eq 1)

Differentiate both sides (eq 1), w.r.t x.

$\Rightarrow dy/dx = d(2x^3)/dx - d(3x^2)/dx + d(6)/dx$

$\Rightarrow dy/dx = 6 \text{ x } 2 - 6x + 0$

$\Rightarrow dy/dx = 6x^2 - 6x$ ------ (eq 2)

Putting $dy/dx = 0$ to find critical points,

$\Rightarrow 6x^2 - 6x = 0$

$\Rightarrow 6x (x - 1) = 0$

$\Rightarrow x = 0,1$

The critical points are 0 & 1.

Differentiate both sides of (eq 2), w.r.t x.

$\Rightarrow d^2y/dx^2 = d(6x^2)/dx - d(6x)/dx$

$\Rightarrow d^2y/dx^2 = 12x - 6$

Now, put the values of x and find the max or min value.

At $x = 0$, $d^2y/dx^2 = 12(0) - 6 = -6 < 0$, hence $x = 0$ is a point of maxima

At $x = 1$, $d^2y/dx^2 = 12(1) - 6 = 6 > 0$, hence $x = 1$ is a point of minima

Answer: The maxima of the function is at $x = 0$ and minima of the function is at $x = 1$.

Example 4: Find the extreme of the given function: $f(x) = -3x^2 + 4x + 7$ and the extremum value using maxima and minima formulas.

Solution:

Using second order derivative test for the maxima and minima of a function:

Given function: $f(x) = -3x^2 + 4x + 7$ ----------------(eq 1)

Differentiate on both sides of (eq 1), w.r.t x.

$\Rightarrow dy/dx = d(-3x^2)/dx + d(4x)/dx + d(7)/dx$

$\Rightarrow dy/dx = - 6x + 4$

Putting $dy/dx = 0$ to find critical points.

$\Rightarrow -6x + 4 = 0$ ----------------(eq 2)

$\Rightarrow x = 2/3$

The critical point is 2/3.

Differentiate both sides of (eq 2), w.r.t x.

$\Rightarrow d^2y/dx^2 = d(-6x)/dx + d(4)/dx$

$\Rightarrow d^2y/dx^2 = -6$

Since $d^2y/dx^2 < 0$, the given curve will have maxima at $x = 2/3$.

The maxima value of f(x) at x = 2/3 is,

$f(2/3) = -3(2/3)^2 + 4(2/3) + 7 = -4/3 + 8/3 + 7 = 25/3$

Answer: The maxima of the function is at x = 2/3 and maximum value is 25/3.

13.6. Summary

- ✔ There can only be one absolute maxima of a function and one absolute minimum of the function over the entire domain.
- ✔ Maxima and minima are the peaks and valleys on the curve of a function.
- ✔ If a function f is either increasing or decreasing in I, it is said to be a monotonous function in I.
- ✔ **Local maxima:** If f'(x) changes sign from positive to negative as x increases via point c, then f(c) gives the maximum value of the function in that range.
- ✔ **Local minima:** If f'(x) changes sign from negative to positive as x increases via point c, then f(c) gives the minimum value of the function in that range.
- ✔ **Point of inflection:** If the sign of f'(x) doesn't change as x increases via c, and the point c is neither the maxima nor minima of the function, then the point c is called the point of inflection.
- ✔ **Local maxima:** If $f''(c) < 0$
- ✔ **Local minima:** If $f''(c) > 0$
- ✔ **Test fails:** If $f''(c) = 0$

13.7. Self-assessment questions

1. What is maxima and minima
2. Explain working procedure of maxima and minima
3. Define local maxima and minima
4. Define absolute maxima and absolute minima
5. Write down properties of maxima and minima
6. Find the maxima and minima for $f(x) = 2x^3-21x^2+36x-15$
7. A stone is thrown in the air. Its height at any time t is given by $h = -5t^2+10t+4$. Find its maximum height.
8. Find the local maxima and minima for the function $y = x^3 – 3x + 2$.
9. How to find out the absolute maxima of a function?
10. Explain second derivative test.

13.8. Reference

- ✔ https://byjus.com/jee/maxima-and-minima-in-calculus/
- ✔ https://www.vedantu.com/maths/maxima-and-minima-of-functions

Unit 14

Theory of Indices

Learning objectives

After studying the unit, students will be able to:

- Use of indices
- Rules of indices
- Concept of logarithm
- Types of logarithm
- Laws of logarithm

Structure

14.1. Introduction

14.2. Theory of Indices

14.3. Laws of Indices Formulas

14.4. Introduction to logarithm

14.5. Logarithms

14.6. Logarithm Types

14.7. Laws of Logarithms

14.8. Summary

14.9. Self-assessment questions

14.10. Reference

14.1. Introduction

The term "index" refers to the exponent or power applied to a number or variable. Indexes is the name for index in the plural. If we write 2^5 or a4, here are the indices 5 and 4. Each number naturally has an index of 1, but since this index does not signify a change in value mathematically, we do not write it. We must record the index as the power of the base number if it is something other than 1. A number's index could potentially be zero or negative.

The index shows how many times a number must be divided by itself. Several index principles that we will describe here regulate these figures.

$a^n = a* a* a* a\cdot\ \cdot\ \cdot\ \cdot$ (n times)

Here,a is the base and n is termed as the index.

$2^5 =$

$2* 2* 2* 2* 2= 32$

$10^4=$

$10* 10* 10* 10 = 10000$

As per indices definition, a number or a variable may have an index. It tells us about how many times the base number is to be multiplied by itself.

14.2. Theory of Indices

A collection of fundamental guidelines that define how indexes or indices are to be treated mathematically are known as the laws of indices. These indices rules are extremely important because indices have a purpose other than simply making it easier to write numbers numerically.

You can only solve the algebraic indices puzzles once you are familiar with the Laws of Indices.

We will go over each law of indices formula with an example of an index law for each type of algebraic index.

14.3. Laws of Indices Formulas

All of the indices laws that you will come across when working with indices are listed below. These are all the fundamental laws that control the indices rules, regardless of how complex the issue is.

1. **Multiplication**
    a. If two terms with a similar base are to be multiplied by each other, the indices have to be added.
    b. $a^n . a^m = a^{n+m}$
    c. **Example:**
    d. $2^3 . 2^6 = 2^{3+6} = 2^9$
2. **Division**
    a. If two terms with a similar base are to be divided, the indices have to be subtracted
    b. $a^n a^m = a^{n-m}$
    c. **Example:**
    d. $4^6/4^4 = 4^{6-4}$
3. **Power of a Power**
    a. If the index of a number is itself raised into another power, then the two indices have to be multiplied.
    b. $(a^n)^m = (a^{nm})$
    c. **Example:**
    d. $(3^3)^4 = (3^{12})$
4. **Negative Power**
    a. If a term has a negative index it can be represented as reciprocal with the positive index as its power.
    b. $(a^{-n}) = 1/a^n$
    c. **Example:**
5. $(5^{-2}) = 1/5^2$

6. **Zero Power**
    a. If a term has the index as 0, then the value of the term becomes one, no matter what the base value is.
7. $a^0 = 1$
    a. **Example:**
    b. $7^0 = 1$

8. **Multiplication with Similar Indices and Different Base**
    a. If two terms in multiplication with each other have similar indices but different bases, then the two bases are multiplied with each other.
9. $a^n . b^n = (ab)^n$
    a. **Example:** $4^2 . 3^2 = (12)^2$
10. **Division with Similar Indices and Different Base**

    If two terms in a division with each other have similar indices but different bases, then the two bases are to be divided with each other.

    $a^n/b^n = (a/b)n$

    Example: $3^2/ 2^2 = (3/2)^2$

14.4. Introduction to logarithm

The other technique to write exponents in mathematics is using logarithms. A number's base-based logarithm is equal to another number. Exponentiation's opposing function is a logarithm.

For example, if $10^2 = 100$ then $\log_{10} 100 = 2$.

Hence, we can conclude that,

$\mathbf{Log_b x = n}$ **or** $\mathbf{b^n = x}$

Where b is the base of the logarithmic function.

This can be read as “Logarithm of x to the base b is equal to n”.

History

John Napier introduced the concept of Logarithms in the 17th century. Later it was used by many scientists, navigators, engineers, etc for performing various calculations which made it simple. In simple words, Logarithms are the inverse process of exponentiation.

14.5 Logarithms

The power to which a number must be increased in order to obtain additional values is referred to as a logarithm. The easiest approach to express enormous numbers is this manner. Numerous significant characteristics of a logarithm demonstrate that addition and subtraction logarithms can also be represented as multiplication and division of logarithms.

“The logarithm of a positive real number a with respect to base b, a positive real number not equal to 1[nb 1], is the exponent by which b must be raised to yield a”.

i.e. $\mathbf{b^y = a \Leftrightarrow \log_b a = y}$

Where,

- “a” and “b” are two positive real numbers
- y is a real number
- “a” is called argument, which is inside the log
- “b” is called the base, which is at the bottom of the log.

In other words, the logarithm gives the answer to the question “How many times a number is multiplied to get the other number?”.

For example, how many 2’s are multiplied to get the answer 32?

If we multiply 2 for 5 times, we get the answer 32.

Therefore, the logarithm is 3.

The logarithm form is written as follows:

$\text{Log}_2 (32) = 5$….(1)

Therefore, the base 2 logarithm of 32 is 5.

The above logarithm form can also be written as:

2x2x2 x2x2 = 32

$2^5 = 32$ …..(2)

Thus, the equations (1) and (2) both represent the same meaning.

Below are some of the examples of conversion from exponential forms to logarithms.

Exponents	**Logarithms**
$6^3 = 216$	$\text{Log}_6\ 216 = 3$
$10^3 = 1000$	$\text{Log}_{10}\ 1000 = 3$
$3^3 = 27$	$\text{Log}_3\ 27 = 3$

14.6. Logarithm Types

In most cases, we always deal with two different types of logarithms, namely

- Common Logarithm
- Natural Logarithm

Common Logarithm

The common logarithm is also called the base 10 logarithms. It is represented as log10 or simply log. For example, the common logarithm of 1000 is written as a log (1000). The common logarithm defines how many times we have to multiply the number 10, to get the required output.

For example, log (100) = 2

If we multiply the number 10 twice, we get the result 100.

Natural Logarithm

The natural logarithm is called the base e logarithm. The natural logarithm is represented as ln or loge. Here, “e” represents the Euler’s constant which is approximately equal to 2.71828. For example, the natural logarithm of 78 is written as ln 78. The natural logarithm defines how many we have to multiply “e” to get the required output.

For example, ln (78) = 4.357.

Thus, the base e logarithm of 78 is equal to 4.357.

14.7. Laws of Logarithms

There are certain rules based on which logarithmic operations can be performed. The names of these rules are:

- Product rule
- Division rule
- Power rule/Exponential Rule
- Change of base rule
- Base switch rule
- Derivative of log
- Integral of log

Using the Indices rules, we can formulate the laws of indices and logarithms.

1. **Multiplication**

   $log_b(x \,.\, y) = log_b(x) + log_b(y)$

   Example:

   $log_{10}(4 \,.\, 3) = log_{10}(4) + log_{10}(3)$

0. **Division**

   $log_b xy = log_b(x) - log_b(y)$

   Example:

   $log_{10}\ (4/3) = log_{10}(4) - log_{10}(3)$

0. **Power of Power**

   $log_b x^m = m.\ log_b(x)$

   Example:

   $log_{10} 2^3 = 3\ log_{10}(2)$

0. **Zero Power** $log_b 1 = 0$

$1 = b^x$, then x=0.

0. **Negative power** log_b **(**$1/x) = - log_b(x)$

Example: $log_{10}1/2 = - log_{10}(2)$

0. Singular Index

logb= 1

Example:

*log*10= 1

7. Base Switch Rule

$\mathbf{log_b (a) = 1 / log_a (b)}$

Example: $\log_b 8 = 1/\log_8 b$

8. Derivative of log

If $f(x) = \log_b(x)$, then the derivative of f(x) is given by;

f'(x) = 1/(x ln(b))

Example: Given, $f(x) = \log_{10}(x)$

Then, f'(x) = 1/(x ln(10))

9. Integral of Log

$\int \log_b(x)dx = x(\log_b(x) – 1/\ln(b)) + C$

Example: $\int \log_{10}(x)\, dx = x \cdot (\log_{10}(x) – 1 / \ln(10)) + C$

Other Properties

Some other properties of logarithmic functions are:

- $Log_b\ b = 1$
- $Log_b\ 1 = 0$
- $Log_b\ 0$ = undefined
-

Logarithms Examples

Example 1:

Solve $\log_2 (32)$ =?

Solution:

since $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$, 5 is the exponent value and $\log_2 (32) = 5$.

Example 2:

What is the value of $\log_{10}(1000)$?

Solution:

In this case, 10^3yields you 1000. So, 3 is the exponent value, and the value of $\log_{10}$ (1000) = 3

Example 3:

Use of the property of logarithms, solve for the value of x for $\log_2 x= \log_2 4+ \log_2 5$

Solution:

By the addition rule, $\log_3 4+ \log_3 5= \log_3 (4 * 5)$

$\text{Log}_3 (20)$. Thus, x= 20.

Example 4:

Solve for x in $\log_2 x = 4$

Solution:

This logarithmic function can be written In the exponential form as $2^4 = x$

Therefore, $2^4= 2 \times 2 \times 2 \times 2 = 16$, x= 316.

Example 5:

Find the value of $\log_5 (1/125)$.

Solution:

Given: $\log_5 (1/125)$

By using the property,

$\text{Log}_b (m/n)=\log_b m – \log_b n$

$\log_5 (1/125) = \log_5 1 – \log_5 125$

$\log_5 (1/125) = 0 – \log_5 5^3$

$\log_5 (1/125) = -3\log_5 5$

$\log_5 (1/25) = -3 (1)$ [By using the property $\log_a a = 1$)

$\log_5 (1/125) = -3$.

Hence, the value of $\log_5 (1/125) = -3$

14.8. Summary

- $\log_b(mn) = \log_b(m) + \log_b(n)$
- $\log_b(m/n) = \log_b (m) – \log_b (n)$
- $\text{Log}_b (xy) = y \log_b(x)$
- $\text{Log}_b m\sqrt{}n = \log_b n/m$

- $m \log_b(x) + n \log_b(y) = \log_b(x^m y^n)$
- $\log_b(m+n) = \log_b m + \log_b(1+nm)$
- $\log_b(m - n) = \log_b m + \log_b (1-n/m)$

14.9. Self-assessment questions

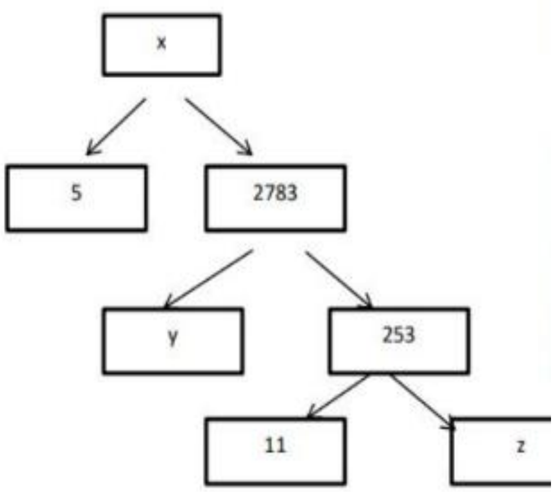
1. Explain theory of indices
2. Write all rule of logarithm
3. Write rule of indices
4. What are types of logarithm?
5. If log x = m+n and log y=m-n, express the value of log $10x/y^2$ in terms of m and n.
6. Express $3^{-2}=1/9$ in logarithmic form.
7. Express log 100.01=-2 in exponential form.
8. Find the logarithm of 1/81 to the base 27.
9. Find x if $\log_7(2x^2-1)=2$.
10. Solve for x, if (log 225/log15) = log x

14.10. Reference

- ✔ https://www.vedantu.com/maths/laws-of-indices
- ✔ https://byjus.com/maths/logarithms/

1. Case Study

In your school, a mathematics exhibition is being held, and one of your pals is building a model of a factor tree. He requests your assistance in completing a quiz for the audience since he is having some trouble.



Consider the factor tree below and respond to the following inquiries.

1. What will be the value of x?
   ANS. 13915

2. What will be the value of y?
   ANS. 11

3. What will be the value of z?
   ANS. 23

4. 13915 is a number that, according on the Fundamental Theorem of Arithmetic, isa
   ANS. Composite Number

5. The Prime Factorisation of 13915 is?
   ANS. 5*112*23

2. Case Study

A manufacturer produces three stationery products

Pencil, Eraser and Sharpener which he sells in two markets. Annual sales are indicated below

Market	Products (in numbers)		
	Pencil	Eraser	Sharpener
A	10,000	2000	18,000
B	6,000	20,000	8,000

If the unit Sale price of Pencil, Eraser and Sharpener are Rs. 2.50, Rs. 1.50 and Rs. 1.00 respectively, and unit cost of the above three commodities are Rs. 2.00, Rs. 1.00 and Rs. 0.50 respectively, then, Based on the above information answer the following:

Question 1: Total revenue of market A

Question 2: Total revenue of Market B

ANSWER : Let the sales of Pencil, Eraser and Sharpener be denoted by matrix

$$\begin{array}{cccccc} & \text{Pencil} & \text{Eraser} & \text{Sharpener} & & \\ X = & \Big[\; 10{,}000 & 2{,}000 & 18{,}000 \; \Big] & \text{Market A} \\ & \; 6{,}000 & 20{,}000 & 8{,}000 & & \text{Market B} \end{array}$$

$$X = \begin{bmatrix} 10{,}000 & 2{,}000 & 18{,}000 \\ 6{,}000 & 20{,}000 & 8{,}000 \end{bmatrix} \begin{matrix} \text{Market A} \\ \text{Market B} \end{matrix}$$

Let the unit sale price of Pencil, Eraser and Sharpener be denoted by matrix Y

$$\text{Unit sale price}$$

$$\text{Let } Y = \begin{bmatrix} 2.50 \\ 1.50 \\ 1.00 \end{bmatrix} \begin{matrix} \text{Pencil} \\ \text{Eraser} \\ \text{Sharpener} \end{matrix}$$

Now,

Total Revenue = Total sales × Unit sales price

= XY

$$= \begin{bmatrix} 10,000 & 2,000 & 18,000 \\ 6,000 & 20,000 & 8,000 \end{bmatrix}_{2\times3} \begin{bmatrix} 2.50 \\ 1.50 \\ 1.00 \end{bmatrix}_{3\times1}$$

$$= \begin{bmatrix} 10,000(2.50) + 2,000(1.50) + 18,000(1) \\ 6,000(2.50) + 20,000(1.50) + 8,000(1) \end{bmatrix}$$

$$= \begin{bmatrix} 25,000 + 3,000 + 18,000 \\ 15,000 + 30,000 + 8,000 \end{bmatrix} = \begin{bmatrix} 46,000 \\ 53,000 \end{bmatrix}$$

$$\text{Total Revenue} = \begin{bmatrix} 46,000 \\ 53,000 \end{bmatrix} \begin{matrix} \longrightarrow \text{Market A} \\ \longrightarrow \text{Market B} \end{matrix}$$

Hence,

Total revenue of Market A = Rs. 46,000

Total revenue of Market B = Rs. 53,000

ANSWER A : Rs 46,000

ANSWER B : Rs 53,000

3.Case Study

If a real valued function f(x) is finitely derivable at any point of its domain, it is necessarily continuous at that point. But its converse need not be true.

For example, every polynomial, constant function is both continuous as well as differentiable and inverse trigonometric functions are continuous and differentiable in its domains etc.

Based on the above information, answer the following questions.

1 If $f(x) = \begin{cases} x, \text{ for } x \le 0 \\ 0, \text{ for } x > 0 \end{cases}$, then at $x = 0$

ANSWER : f(x) is continuous and differentiable

2 If $f(x) = |x - 1|, x \in R$, then at $x = 1$

ANSWER : f(x) is continuous but not differentiable

3 f(x) = l sin x l, then which of the following is true?

ANSWER : F(x) is discontinuous at x=0

4. Case Study

A general election of Lok Sabha is a gigantic exercise. About 911 million people were eligible to vote and voter turnout was about 67%, the highest ever Let I be the set of all citizens of India who were eligible to exercise their voting right in general election held in 2019. A relation 'R' is defined on I as follows: R = {(□1, □2) ∶ □1, □2 ∈ □ and both use their voting right in general election – 2019}

1. Two neighbors X and Y∈ I. X exercised his voting right while Y did not cast her vote in the general election – 2019. Which of the following is true?

a. (X,Y) ∈ R b. (Y,X) ∈ R

c. (X,X) ∉ R d. (X,Y) ∉ R

2. Mr.'□' and his wife '□'both exercised their voting right in general election -2019, Which of the following is true?

a. both (X,W) and (W,X) ∈ R b. (X,W) ∈ R but (W,X) ∉ R

c. both (X,W) and (W,X) ∉ R d. (W,X) ∈ R but (X,W) ∉ R

3. Three friends F1, F2 and F3 exercised their voting right in general election-2019, then Which of the following is true?

a. (F1,F2) ∈ R, (F2,F3) ∈ R and (F1,F3) ∈ R

b. (F1,F2) ∈ R, (F2,F3) ∈ R and (F1,F3) ∉ R

c. (F1,F2) ∈ R, (F2,F2) ∈ R but (F3,F3) ∉ R

d. (F1,F2) ∉ R, (F2,F3) ∉ R and (F1,F3) ∉ R

4. The above defined relation R is __________

a. Symmetric and transitive but not reflexive

b. Universal relation

c. Equivalence relation

d. Reflexive but not symmetric and transitive

5. Mr. Shyam exercised his voting right in General Election – 2019, then Mr. Shyam is related to which of the following?

a. All those eligible voters who cast their votes

b. Family members of Mr.Shyam

c. All citizens of India

d. Eligible voters of India

ANSWERS

1. (d) (X,Y) ∉ R

2. (a) both (X,W) and (W,X) ∈ R

3. (a) (F1,F2) ∈ R, (F2,F3) ∈ R and (F1,F3) ∈ R

4. (c) Equivalence relation

5. (a) All those eligible voters who cast their votes

5. Case Study

Two farmers Ramakishan and Gurucharan Singh cultivate only three varieties of rice namely Basmati, Permal and Naura. The sale (in rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B

September sales (in Rupees)

A= [10,000 20,000 30,000
50,000 30,000 10,000] □□□□□□□h□□
□□□□□h□□□□

October sales (in Rupees)

B= [5,000 10,000 6,000
20,000 10,000 10,000] □□□□□□□h□□
□□□□□h□□□□

1. The total sales in September and October for each farmer in each variety can be represented as _______..

a. A+B

b. A-B

c. A>□

d. A<□

2. What is the value of □23?

a. 10000

b. 20000

c. 30000

d. 40000

3. The decrease in sales from September to October is given by _______ .

a. A+B

b. A-B

c. A>□

d. A<□

4. If Ramkishan receives 2% profit on gross sales, compute his profit for each variety sold in October.

a. Rs. 100, Rs. 200 and Rs. 120

b. Rs. 100, Rs. 200 and Rs. 130

c. Rs. 100, Rs. 220 and Rs. 120

d. Rs. 110, Rs. 200 and Rs. 120

5. If Gurucharan receives 2% profit on gross sales, compute his profit for each variety sold in September.

a. Rs. 100, Rs. 200, Rs. 120

b. Rs. 1000 , Rs. 600, Rs. 200

c. Rs. 400, Rs. 200, Rs. 120

d. Rs. 1200, Rs. 200, Rs. 120

ANSWERS

1.(a) A+B

2. (a) 10000

3. (b) A-B

4. (a) Rs. 100, Rs. 200 and Rs. 120

5. (b) Rs. 1000, Rs. 600, Rs. 200